UNIT : 1

PRODUCT OF VECTORS AND VECTOR DIFFERENTIATION

UNIT- 1

BLOCK INTRODUCTION

In previous courses students have already been studied the concepts of dot and cross product of two vectors. In the present block initially a brief review of basic concepts and some useful formulae are given to recall the matter. In the second phase of this block we introduce the concepts of product of three and four vectors in a simple manner followed by illustrated examples.

Object :- At the end of this block the reader would be able to understand the said concept.

UNIT : 1

Unit-I : Product of Three & Four Vectors And Vector Differentiation

STRUCTURE

1.0 Introduction

In Physics and Geometry we generally deal with the different physical quantities e.g. length, mass, volume, density, weight, velocity, force etc. According to the basic properties of these quantities, they are classified into two types viz. scalar and vector quantities.

Physical quantities which have only magnitude but have no definite direction are defined as scalars while those having both magnitude and direction defined as vectors.

1.1 Brief Review

We know :

(1) $\vec{a}.\vec{b} = ab\cos\theta$, a scalar quantity .

(2) $\vec{a}\times\vec{b} = ab\sin\theta.\hat{n}$, a vector quantity where $\hat{n}$ is a unit vector $\perp$ to the plane of vectors $\vec{a}$ and $\vec{b}$.

(3) If $|\vec{a}| = 1$,then $\vec{a}$ is called unit vector.

(4) $\vec{a}.\vec{b}. = \vec{b}.\vec{a}$ i.e. dot product is commutative

(5) $\vec{b}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$

(6) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

(7) $\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$

(8) Two vectors are said to be mutually perpendicular if $\vec{a}.\vec{b} = 0$

(9) Two vectors are said to be parallel or collinear if $\vec{a} = \lambda\vec{b}$ where λ is a scalar.

(10) If $\vec{a}\times\vec{b} = 0$ then vectors $\vec{a}$ and $\vec{b}$ are parallel. Also $\vec{a}\times\vec{a} = 0$ i.e. [every vector is parallel to itself]

(11) $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$

(12) $\vec{i}\times\vec{j}=\vec{k}, \vec{j}\times\vec{k}=\vec{i}, \vec{k}\times\vec{i}=\vec{j}$

(13) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ but $\vec{b}\times\vec{a}=-(\vec{a}\times\vec{b})$

(14) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

(15) If $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b}=b_1i+b_2j+b_3k$ then $\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$

and

$$\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1.2 SCALAR TRIPLE PRODUCT: The product of two vectors one of which is itself the vector product of two vectors is a scalar quantity called scalar triple product. In short if $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then $\vec{a}.(\vec{b}\times\vec{c})$ is called scalar triple product. Symbolically, it is denoted by [abc]

THINGS TO REMEMBER

NOTE-1 $(\vec{a}.\vec{b})\times\vec{c}$ is meaningless since $(a.b)$ is not a vector quantity.

NOTE-2 $\vec{a}.(\vec{b}\times\vec{c})=(\vec{b}\times\vec{c}).\vec{a}$

NOTE-3 $\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}$

NOTE-4 Volume with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ of a parallelopiped $=\vec{a}.(\vec{b}\times\vec{c})=[abc]$

NOTE-5 Scalar triple product of vectors $\vec{a}, \vec{b}, \vec{c}$ also represent as [abc] =[bca] = [cab]

NOTE-6 [abc] = –[bac] = – [acb] i.e. sign is changed provided the cycle order be changed .

NOTE-7 $[ijk]=\vec{i}.(\vec{j}\times\vec{k})=\vec{i}.\vec{i}=1$

NOTE-8 Three vectors $\vec{a}, \vec{b}, \vec{c}$ are said to be coplanar if $[\vec{a}, \vec{b}, \vec{c}] = 0$.

NOTE-9 Determinant form of scalar triple product of vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ and $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ is given by

$$[abc] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

NOTE-10 Scalar triple product [abc] = 0 when two of them are equal i.e. [aac] =[abb] = [aba] etc. =0

SOLVED EXAMPLES

EX.1 If $\vec{a} = 2\vec{i} + \vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{c} = 3\vec{i} + \vec{j} + 2\vec{k}$, find $\vec{a}.(\vec{b} \times \vec{c})$

Sol**.** Here

$$(\vec{b} \times \vec{c}) = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3\vec{i} + 5\vec{j} - 7\vec{k}$$

therefore $\vec{a}.(\vec{b} \times \vec{c})$ = $(2i+j+3k).(3i+5j-7k)$

= $2.3 + 1.5 + 3.(-7)$ $\because i.j = 0$ etc.

= 6+5-21 $\because i^2 = 1 = j^2 = k^2$

= –10. **Ans.**

Alternative method :

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$= 2(4 – 1) – 1 (–2–3) + 3(–)–6)$

$= 6 + 5 – 21$

$= –10$ **Ans.**

Ex.2 Show that the vectors $\vec{a}=\vec{i}+3\vec{j}+\vec{k}$, $\vec{b}=-2\vec{i}-\vec{j}-\vec{k}$ and $\vec{c}=7\vec{j}+3\vec{k}$ are parallel to the same plane.

Sol. $[abc]=\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & 7 & 3 \end{vmatrix}=1(-3+7)-2(9-7)=4-4=0$ [See note 10]

so the vectors are coplanar i.e. parallel to the same plane .

EX 3 Find the volume of a parallelopiped whose edges are represented by $\vec{a}=2\vec{i}-3\vec{j}+4\vec{k}$, $\vec{b}=\vec{i}+2\vec{j}-\vec{k}$ and $\vec{c}=3\vec{i}-\vec{j}+2\vec{k}$.

Sol. The required volume V= [$\vec{a}\ \vec{b}\ \vec{c}$]

$$\vec{a}.(\vec{b}\times\vec{c})=\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}=2(4-1)+3(2-3)+4(1-6)=6+15-28=-7=7$$

EX 4. Show that three points $a-2b+3c, -2a+3b-4c and a-3b+5c$ are coplanar .

Sol. Let $\vec{A} = a-2b+3c$, $\vec{B} = -2a+3b-4c$, $\vec{C} = a-3b+5c$

Then $[[\vec{A}\vec{B}\vec{C}]=\vec{A}.(\vec{B}\times\vec{C})$

$= (a-2b+3c)$.[$(-2a+3b-4c)$x$(a-3b+5c)$]

$= (a-2b+3c)$.[6$(a\times b)$-10 $(a\times c)$+3$(b\times a)$+15$(b\times c)-4(c\times a)+12(c\times b)$]

$= (a-2b+3c)$. [3$(a\times b)$+6$(c\times a)$+3$(b\times c)$] {see note 13(1.1)}

$=3[aab]+6[aca]+3[abc]-6[bab]-12[bca]-6[bbc]+9[cab]+18[cca]+9[cbc]$

$=3[abc]-12[bca]+9[cab]$ {see note 10}

The vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar .

EX 5. Prove that $[a+b, b+c, c+a] = 2[abc]$

Soln. LHS. $=[a+b, b+c, c+a]$

$= (a+b) . [(b+c)\times(c+a)$

$=(a+b).[b\times c+b\times a+c\times a] \because c\times c=0$

$=a.(b\times c)+a.(b\times a)+a.(c\times a)+b.(b\times c)+b.(b\times a)+b.(c\times a)$

$=[abc]+[aba]+[aca]+[bbc]+[bba]+[bca]$

$=[abc]+[bca]$ {note 10}

$=[abc]+[abc]$

$= 2\ [abc]$ Hence Proved.

CHECK YOUR PROGRESS :

Q.1 If $\vec{a} = -2i - 2j + 4k, b = -2i + 4j - 2k$ and $\vec{c} = 4i - 2j - 2k$ then evaluate a.(b×c) and interpret the result [Ans 0]

Q. 2 Find the value of λ so that the vectors $2\vec{i}+\vec{j}+\vec{k}$, $\vec{i}+2\vec{j}+3\vec{k}$ and $3\vec{i}+\lambda\vec{j}+5\vec{k}$ are coplanar. [Ans. -4]

Q .3 Show that if $\vec{a}, \vec{b}, \vec{c}$ are non coplanar than $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also non coplanar. Is this true for $\vec{a}-\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$?

Q.4 If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors then show that [axb bxc cxa] = $[abc]^2$.

Q. 5 Show that the four points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if [bcd]+[cad]+[abd]=[abc].

1.3 VECTOR TRIPLE PRODUCT :

Definition:- The vector product of two vectors one of which is itself the vector product of two vectors is a vector quantity called vector triple product. In short if $\vec{a},\vec{b},\vec{c}$ are three vectors , then $\vec{a}\times\left(\vec{b}\times\vec{c}\right) and \left(\vec{a}\times\vec{b}\right)\times\vec{c}$ are called vector triple product .

THINGS TO REMEMBER :

Note 1. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}.\vec{c})\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

Way to remember : $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$= (outer . remote) adjacent – (outer . adjacent) remote

Where a=outer, b= adjacent , c= remote.

Note 2. $\left(\vec{b}\times\vec{c}\right)\times\vec{a}=-\vec{a}\times\left(\vec{b}\times\vec{c}\right)$= - [(a.c)b-(a.b)c] = (b.a)c – (c.a)b

$-[(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}]=(\vec{b}.\vec{a})\vec{c}-(\vec{c}.\vec{a})\vec{b}$

SOLVED EXAMPLES

Ex. 1 Verify that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$, given that

$\vec{a}=i+2j+3k\ ,\vec{b}=2i-j+k$ and $\vec{c}=3i+2j-5k$

Soln . Here

$$\vec{b}\times\vec{c}=\begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 2 & 5 \end{vmatrix}=3i+13i+7k \text{ therefore}$$

$$\therefore \vec{a}\times\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 13 & 7 \end{vmatrix}=-25i+2j+7k \qquad \text{..................(1)}$$

Now $\vec{a}.\vec{c}=(\ i+2j+3k\ ).(3i+2j-5k)=-8$

similarly $\vec{a}.\vec{b}=(i+2j+3k).(2i-j+k\ )=3$

$\therefore \quad (\vec{a}.\vec{c})\vec{b}\ \ (\vec{a}.\vec{b})\vec{c}=-8(2i-j+k\ )-3(3i+2j-5k)$

$$=-25i+2j+7k \qquad \text{............(2)}$$

from (1) and (2)

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\,\vec{b}-(\vec{a}.\vec{b})\,\vec{c}$ verified

Ex. 2 Prove that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$

Soln. LHS. $=\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\,\vec{b})$

$=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{a})\vec{c}+(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}$

= 0 [since $\vec{a}.\vec{b}=\vec{b}.\vec{a}$ etc.]

Ex. 3 Prove that $\vec{i}\times(\vec{a}\times\vec{i})+\vec{j}\times(\vec{a}\times\vec{j})+\vec{k}\times(\vec{a}\times\vec{k})=2\vec{a}$

Sol. We know that

$\vec{i}\times(\vec{a}\times\vec{i})=(\vec{i}.\vec{i})\vec{a}-(\vec{i}.\vec{a})\vec{i}\times=\vec{a}-a_1\vec{i}$

$[\vec{i}.\vec{i}=1$ also if $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ then $\vec{a}.\vec{i}=a_1$] ....(1)

similarly $\vec{j}\ (\vec{a}\ \vec{j})=\vec{a}-a_2\vec{j}$ ……….(2)

$\vec{k}\ (\vec{a}\ \vec{k})=\vec{a}-a_3\vec{k}$ ……….(3)

on adding (1),(2) & (3) we get

$=3\vec{a}\ \ \vec{a}=2\vec{a}$.

Ex. 4 Prove that $[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times a]=[abc]^2$

Soln. Let $\vec{b}\times\vec{c}=d$

Then $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=\vec{d}\times(\vec{c}\times\vec{a})$

$=(\vec{d}.\vec{a})\vec{c}-(\vec{d}.\vec{c})\vec{a}$

$=[(\vec{b}\times\vec{c}.\vec{a})]\vec{c}-[(\vec{b}\times\vec{c}.\vec{c})]a$

$==[bca]\vec{c}-[bcc]\vec{c}$

$=[bca]\vec{c}$ since $[bcc]=0$.(2)

Now [axb bxc cxa] $=(\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c})\ \ (\vec{c}\times\vec{a})]$

$=(\vec{a}\times\vec{b}).[bca]\vec{c}$ [by (2)]

$= [bca]\,(\vec{a}\times\vec{b}).\vec{c}$

$= [bca]\,[abc] = [abc]^2$ $\qquad [[\because [bca]=[abc]]$

Hence proved .

1.4 SCALAR PRODUCT OF FOUR VECTORS :

Definition:- Let $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are four vectors , then $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ is a scalar product of four vectors and in brief it is defined as

$$\left(\vec{a}\times\vec{b}\right).\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}=(a.c)(b.d)-(a.d)(b.c)$$

1.5 VECTOR PRODUCT OF FOUR VECTORS :

Definition :- Let $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are four vectors , then $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is a scalar product of four vectors and it defined in brief as

$$(\vec{a}\times\vec{b})\ \ (\vec{c}\times\vec{d}) = [abd]c-[abc]d \quad = [acd]b-[bcd]a$$

way to remember : product = c [remaining] – d [remaining].

SOLVED EXAMPLES

Ex.1 If $\vec{a}=\vec{i}+2\vec{j}-\vec{k}\ ,\vec{b}=3\vec{i}-4\vec{k}\ ,\vec{c}=-\vec{i}+\vec{j}$ and $\vec{d}=2\vec{i}-\vec{j}+3\vec{k}$ then find

(i) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$ (ii)$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

Soln (i) $$\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & 0 & -4 \end{vmatrix}=-8\vec{i}+\vec{j}+6\vec{k}$$

Similarly $$\vec{c}\times\vec{d}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix}=3\vec{i}+3\vec{j}-\vec{k}$$

now $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (-8i+j-6k).(3i+3j-k) = -24+3+6 = -15$ Ans.

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ = [abd]c - [abc]d

now $[abd] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & -4 \\ 2 & -1 & 3 \end{vmatrix} = -4-34+3 = -35$

and $[adc] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & -4 \\ -1 & 1 & 0 \end{vmatrix} = 4+8-3 = 9$

$\because (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -35(-i+j) - 9(2i-j+3k) = 17\vec{i} - 26\vec{j} - 27\vec{k}$ Ans.

Ex. 2 If the vectors a ,b, c, d are coplanar then show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0$.

Solv. We know that $(\vec{a}\times\vec{b})$ is a vector perpendicular to the plane of a and b . Similarly $(\vec{c}\times\vec{d})$ is perpendicular to the plane of c and d .
But $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar

Hence $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ are perpendicular to the same plane in which $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar.

Therefore $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ are parallel to each other.

Hence $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0$.

Ex. 3 Prove that $2a^2 = |\vec{a}\times\vec{i}|^2 + |\vec{a}\times\vec{j}|^2 + |\vec{a}\times\vec{k}|^2$, where $|\vec{a}|$= a.

Solv. $|\vec{a}\times\vec{i}|^2 = (\vec{a}\times\vec{i}).(\vec{a}\times\vec{i}) = \begin{vmatrix} a.a & a.i \\ i.a & i.i \end{vmatrix} = a^2 - (a.i)^2$ ....(i)

similarly $|\vec{a}\times\vec{i}|^2 = a^2 - (a.j)^2$ ……….(ii)

$|\vec{a}\times\vec{k}|^2 = a^2 - (a.k)^2$ ………….(iii)

On adding (i), (ii), (iii) ,we get

$$\left|\vec{a}\times\vec{i}\right|^2+\left|\vec{a}\times\vec{j}\right|^2+\left|\vec{a}\times\vec{k}\right|^2=3a^2-\left[(a.i)^2+(a.j)^2+(a.k)^2\right]$$

$$= 3a^2 - a^2$$

$$= 2a^2$$ Hence proved.

1.6 RECIPROCAL SYSTEM OF VECTORS :

Definition:- The three vectors a', b', c' defined as

$a'=\vec{b}\times\vec{c}/[abc]$,$b'=\vec{c}\times\vec{a}/[abc]$ and $c'=\vec{a}\times\vec{b}/[abc]$ are called the reciprocal to the vectors $\vec{a},\vec{b},\vec{c}$ which are non-coplanar so that [abc] $\neq$ 0 .

THINGS TO REMEMBER :

Note1 $a.a'=\vec{a}.(\vec{b}\times\vec{c})/[abc]=[abc]/[abc]=1$,similarly $b.b'=1$, $c.c'=1$

Remark : Due to this property the word reciprocal has been used.

Note 2 $a.b'=a.(\vec{c}\times\vec{a})/[abc]=[aca]/[abc]=0$, similarly a.c' =0= b.c' = b.a' = c.a'=c.b'.

Note 3 If a', b', c' are three non-coplanar vectors then [abc] is reciprocal to [a' b' c'] where a',b',c' are the reciprocal vectors .

Remark : $[a'\,b'\,c']=[\frac{(\vec{b}\times\vec{c})}{[abc]},\frac{(\vec{c}\times\vec{a})}{[abc]},\frac{(\vec{a}\times\vec{b})}{[abc]}]$

$$=\frac{1}{[abc]^3}.\{[\vec{b}\times\vec{c},\vec{c}\times\vec{a},\vec{a}\times\vec{b}]\}$$

$$=\frac{1}{[abc]^3}.[abc]^2$$ {by Ex.3 ,1.3}

$$=\frac{1}{[abc]}$$ therefore [a' b' c'] [abc] = 1, as required.

Note 4 The system of unit vectors i, j, k is its own reciprocal, because

$i'=\vec{j}\times\vec{k}/[ijk]=i/1=i$

$\{\text{since } \vec{j}\times\vec{k}=\vec{i}\ \&[i\,j\,k]=1\}$, similarly $j'=j$ and $k'=k$.

SOLVED EXAMPLES

Ex.1 Find a set of vectors reciprocal to the set $2\vec{i} + 3\vec{j} - \vec{k}, \vec{i} - \vec{j} - 2\vec{k}, -\vec{i} + 2\vec{j} + 2\vec{k}.$

Soln Let $\vec{a} = 2i + 3j - k, \vec{b} = i - j - 2k, \vec{c} = -i + 2j + 2k$

We know that by the system of reciprocal vectors as

$a' = \vec{b} \times \vec{c}/[abc]$ $, b' = \vec{c} \times \vec{a}/[abc]$ and $c' = \vec{a} \times \vec{b}/[abc]$

Therefore $\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\vec{i} + \vec{k}$

similarly $\vec{c} \times \vec{a} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -8\vec{i} + 3\vec{j} - 7\vec{k}$

and $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = -7\vec{i} + 3\vec{j} - 5\vec{k}$

also $[abc] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 1 & 2 & 2 \end{vmatrix} = 4 - 1 = 3$

so the required reciprocal vectors are

$a' = (2i + k)/3, b' = (-8i + 3j - 7k)/3, c' = (-7i + 3j - 5k)/3$.

Ex 2 Prove that $(\vec{a} \times a') + (\vec{b} \times b') + (\vec{c} \times c') = 0$

Solv. We know that $a' = (\vec{b} \times \vec{c}) / [abc]$,

therefore $(\vec{a} \times \vec{a}') = [\vec{a} \times (\vec{b} \times \vec{c})] / [abc] = [(a.c)b - (a.b)c] / [abc]$ . ...……(i)

similarly $(\vec{b} \times \vec{b}') = [(b.a)c - (b.c)a] / [abc]$ ……………(ii)

and $(\vec{c} \times \vec{c}') = [(c.b)a - (c.a)b] / [abc]$ ……………(iii) now on adding (i), (ii), (iii),we get

$$(\vec{a}\times\vec{a}')+(\vec{b}\times\vec{b}')+(\vec{c}\times\vec{c}')=0. \quad \text{hence proved.}$$

Ex 3 If $a'=\vec{b}\times\vec{c}\ /\ [abc]$,$b'=\vec{c}\times\vec{a}\ /\ [abc]$ and $c'=\vec{a}\times\vec{b}\ /\ [abc]$, then prove that

$a=(\vec{b}'\times\vec{c}')/[a'b'c'],b=(\vec{c}'\times\vec{a}')/[a'b'c'],c=(\vec{a}'\times\vec{b}')/[a'b'c'].$

Solv. We have $b'\times c' = \{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}/[abc]^2$

$= \{\vec{d}\times(\vec{a}\times\vec{b})\}/[abc]^2$ [where $d=\vec{c}\times\vec{a}$]

$= \{(d.b)a - (d.a)b\} / [abc]^2$ by the def. of vector triple product

$= \{[cab]\ a - [caa]b\} / [abc]^2$ since $d=c\times a$

$= [abc\ ]a / [abc]^2$ since $[caa]=0$

$= a / [abc]$ ………..(1)

now $[a'b'c'] = \vec{a}'.(\vec{b}'\times\vec{c}')$

$=\{(\vec{b}\times\vec{c})/[abc]\}.\{a/[abc]\}$, from{1}

$= (\vec{b}\times\vec{c}).a / [abc]^2$

$= [abc] / [abc]^2 = 1 / [abc]$ ……………..(2)

clearly from (1) & (2) we have

$(\vec{b}'\times\vec{c}')/[a'b'c'] = a$

similarly we can prove the other parts.

CHECK YOUR PROGRESS :

Q.1 Find the value of $\vec{a}\times(\vec{b}\times\vec{c})$ if the vectors are

$\vec{a}=i+2j-2k,\vec{b}=2i-j+k,\vec{c}=i+3j-k. Ans.20i-3j+7k]$

Q.2 Verify that $a\times(b\times c) = (a.c)b-(a.b)c$ where

$\vec{a}=i+2j+3k,\vec{b}=2i-j+k,\vec{c}=3i+2j-5k$ a.

Q.3 Prove that $\vec{a}\times(\vec{b}\times\vec{a})=(\vec{a}\times\vec{b})\times\vec{a}$.

Q.4 Show that $\vec{i}\times(\vec{j}\times\vec{k})=(\vec{i}\times\vec{j})\times\vec{k}=0$.

Q.5 Prove that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0.$

Q.6 Prove that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})=-2[bcd]a$.

Q.7 If a,b,c and a',b',c' are the vectors of reciprocal system, prove that

(i) $a.a'+b.b'+c.c'=3.$

(ii) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=(a+b+c)/[abc],[abc]\neq 0.$

1.7 VECTOR DIFFERENTIATION :

1.7.1 Vector Function :

If a variable vector $\vec{a}$ depends on a scalar variable t in such a way that as t varies in some interval, the vector $\vec{a}$ also varies ,then $\vec{a}$ is called vector function of the scalar t and we write a = a(t) or f(t).

Ex (1) f(t) =at^2i+2atj+btk (2) a(t) = cos ti+ sin tj+tan tk

1.7.2 Differentiation of Vector Function with respect to a scalar :

Let a be a vector function of $\vec{a}$ scalar t . Let $\delta\vec{a}$ be the small increment in a corresponding to the small increment δt in t

Then $\delta\vec{a}=\vec{a}(t+dt)-\vec{a}(t)$

or $\delta\vec{a}/\delta t=\{\vec{a}(t+dt)-a(t)\}/\delta t$

now as $\delta t\to 0$ the vector function $\frac{\delta\vec{a}}{\delta t}$ tends to a limit which is denoted by $\frac{\vec{d}}{dt}$ and is called the derivative of the vector function a w.r.t. t .

i.e. $$\frac{d}{dt}=\lim_{\delta t\to 0}\frac{\vec{a}(t+\delta t)-\vec{a}(t)}{\delta t}$$

1.7.3 Imp Formulae of Differentiation :

(1) $\frac{d}{dt}(k\vec{a})=k\frac{\vec{d}}{dt}$, where k is constant .

(2) $\frac{d}{dt}\left(\vec{a}+\vec{b}\right)=k\frac{d\vec{a}}{dt}+\frac{d\vec{b}}{dt}$

(3) $\frac{d}{dt}\left(\vec{a}.\vec{b}\right)=a.\frac{d\vec{b}}{dt}+b.\frac{d\vec{a}}{dt}$

(4) $\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)-\vec{a}\times\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}\times\vec{b}$

(5) If a be $\vec{a}$ vector function of some scalar t and we have

$\vec{a}=\mathrm{a_x}\vec{\mathrm{i}}+\mathrm{a_y}\vec{\mathrm{j}}+\mathrm{a_z}\vec{\mathrm{k}}$ referred to rectangular axes OX, OY, OZ then

$$\frac{d\vec{a}}{dt}=\frac{da_x}{dt}\vec{i}+\frac{da_y}{dt}\vec{j}+\frac{da_z}{dt}\vec{k}$$

(6) If $\vec{a}$ is a constant vector, then $\frac{d\vec{a}}{dt}=0$

SOLVED EXAMPLES

Ex.1 If $\vec{a}=(\cos nt)\vec{i}+(\sin nt)\vec{j}$, where n is constant , then show that

(i) $\vec{a}\times\frac{d\vec{a}}{dt}=n.\vec{k}$

(ii) $\vec{a}.\frac{d\vec{a}}{dt}=0$

Soln. Given $\vec{a}=\cos nt\,\vec{i}+\sin nt\,\vec{j}$ ……………..(i)

$\therefore\ \frac{da}{dt}=-n\sin nt\,\vec{i}+n\cos nt\,\vec{j}$ ……………..(ii)

now (i) $\vec{a}\times\frac{d\vec{a}}{dt}=\begin{vmatrix} i & j & k \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix}=k\left(n\cos^2 nt+n\sin^2 nt\right)=n.\vec{k}$ hence proved.

(ii) $\vec{a}.\frac{d\vec{a}}{dt}$ = {(cos nt i + sin nt j)}.{ - n sin nt i + n cos nt j }

= - n cos nt .sin nt + n cos nt .sin nt

= 0 Hence proved.

Ex.2 If $\vec{a} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ then find $\frac{d\vec{a}}{dt}, \frac{d^2\vec{a}}{dt^2}, \left|\frac{d\vec{a}}{dt}\right|, \left|\frac{d^2\vec{a}}{dt^2}\right| at\ t = 0$

Soln. Given $\vec{a} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

therefore $\frac{da}{dt} = 2t\vec{i} - \vec{j} + 2\vec{k}$ ………………(i)

now $\frac{d^2\vec{a}}{dt^2} = 2\vec{i} - 0\vec{j} + 0\vec{k} = 2\vec{i}$ …………………..(ii)

also from (i) & (ii)

$$\left|\frac{da}{dt}\right| = \sqrt{\{4t^2 + 1 + 4\}} = \sqrt{(4t^2 + 5)}$$ ………………….(iii)

$$\left|\frac{d^2\overline{a}}{dt^2}\right| = \sqrt{(2)^2} = 2$$ …………………(iv)

Hence from (i) (ii) (iii) &(iv) at t=0 , we have

$$\frac{d\vec{a}}{dt} = -\vec{j} + 2\vec{k},\ \frac{d^2a}{dt^2} = 2\vec{i}\ \left|\frac{da}{dt}\right| = \sqrt{5} \text{ and } \left|\frac{d^2a}{dt^2}\right| = 2.$$ Ans

Ex.3 Show that $\hat{r} \times d\hat{r} = \frac{\vec{r} \times d\vec{r}}{r^2}$, where $\vec{r} = r.\hat{r}$

Soln. Since $\hat{r} = \frac{\vec{r}}{r}$ ....……..(i)

$$\therefore d\hat{r} = \frac{1}{r}d\vec{r} + \vec{r}d\left(\frac{1}{r}\right)$$

$$= \frac{1}{r}d\vec{r} + \vec{r}\left(-\frac{1}{r^2}dr\right)$$

$$\therefore \hat{r} \times d\hat{r} = \hat{r} \times \left(\frac{1}{r}d\vec{r} - \frac{\vec{r}}{r^2}dr\right)$$

$$= \frac{\vec{r}}{r} \times \left(\frac{1}{r}d\vec{r} - \frac{\vec{r}}{r^2}dr\right)$$ [by (i)]

$$= \frac{1}{r^2}(\vec{r} \times d\vec{r}) - \frac{1}{r^3}(\vec{r} \times d\vec{r})dr$$

$$= \frac{1}{r^2}(\vec{r} \times d\vec{r})\ [\because \vec{r} \times \vec{r} = 0] henceproved$$

Ex.4 If $r = \cos$ it + $a\sin t$ j + $a\tan\alpha$ t k, then find $\left|dr/dt \times d^2r/dt^2\right|$ and

$\left[dr/dt, d^2r/dt^2, d^3r/dt^3\right]$

Solv. Given $r = a\cos ti + a\sin tj + a\tan\alpha tk,$

Therefore $dr/dt = -a\sin ti + a\cos tj + a\tan\alpha k$ …………… (i)

$$d^2r/dt^2 = -a\cos ti - a\sin tj + 0k = -a\cos ti - a\sin tj, \quad \text{……………(ii)}$$

and $d^3r/dt^3 = a\sin ti - a\cos tj$ …………….(iii)

now $$(dr/dt \times d^2r/dt^2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$a^2 \sin t \tan\alpha i - a^2 \cos t \tan\alpha j + a^2 k$$

therefore $\left|dr/dt \times d^2r/dt^2\right| = \sqrt{\left[a^4 \sin^2 t \tan^2\alpha + a^4 \cos^2 t \tan^2\alpha + a^4\right]}$

$$= \sqrt{\left[a^4 \tan^2\alpha + a^4\right]} = a^2 \sec\alpha \qquad \text{Ans}$$

$$\left[\frac{dr}{dt}\,\frac{d^2r}{dt^2}\,\frac{d^3r}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= a\tan\alpha\left[a^2\cos^2 t + a^2\sin^2 t\right] = a^3\tan\alpha \quad \textbf{Ans}$$

Ex.5 If $\vec{a}$ = a(t) has a constant magnitude, then show that $\vec{a} \cdot {d\vec{a}}/{dt} = 0$.

Soln. Given $\vec{a}$ has a constant magnitude ,therefore $|a|^2 = \vec{a}.\vec{a}$ = constant ……(i)

On differentiating (i) both side w.r.t. t , we get

$$d(\vec{a}.\vec{a})/dt = 0$$

⇨ $\vec{a}.\ d\vec{a}/dt + d\vec{a}/dt . \vec{a} = 0$

⇨ $2\,\vec{a}.\ d\vec{a}/dt = 0$

⇨ $\vec{a}.\ d\vec{a}/dt = 0$.

CHECK YOUR PROGRESS :

Q.1 If $r = \vec{a}\cos\omega t + \vec{b}\sin\omega t,$ then show that (i)

$\vec{r} \times d\vec{r}/dt = \omega(\vec{a} \times \vec{b}), (ii) d^2r/dt^2 = -\omega^2.r.$

Q.2 If $\vec{r} = \vec{a}\sinh nt + \vec{b}\cosh nt$, where $\vec{a}$ *and* $\vec{b}$ are constant vectors, then show that $d^2r/dt^2 = n^2$ r.

Q.3 If $\vec{a} = t^2 i \ tj + (2t+1)k$ *and* $\vec{b} = (2t-3)i + j - tk,$ then find

$d(\vec{a}.\vec{b})/dt$, at $t = 1$ [Ans.6]

Q.4 Evaluate (i) d [abc]/dt (ii) d/dt [a, da/dt, d^2a/dt^2]

Q.5 If r be the position vector of P. Find the velocity and acceleration of P at t= $\pi/6$ where $r = \sec t i + \tan t\ j$ {Ans v = (2/3)i+(4/3)j , a= 2(5i+4j)/3√3}

[Hint: velocity=dr/dt, acceleration = d^2r/dt^2]

1.8 Gradiant, Divergence & Curl :

1.8.1 Partial Derivative of Vector :

Let $\vec{F}(x,y)$ be a vector function of independent variable x and y . Then partial derivatives of $\vec{F}$ w.r.t. x and y are denoted by Fx and Fy resp. & defined as

$\vec{F}x = \delta\vec{F}/\delta x = Lt.\delta x \to 0\{[\vec{F}(x+\delta x, y)] - \vec{F}(x,y)\}/\delta x$ And

$$\vec{F}y = \delta\vec{F}/\delta y = Lt.\delta y \to 0\{[\vec{F}(x, y+\delta y)] - \vec{F}(x,y)\}/\delta y$$

Second order partial derivatives are denoted by $\delta^2F/\delta x^2$, $\delta 2F/\delta x \delta y$, $\delta^2F/\delta y^2$. means

$\delta^2F/\delta x^2 = \delta(\delta F/\delta x)/\delta x, \delta^2F/\delta x \delta y = \delta(\delta F/\delta y)/\delta x, \delta^2F/\delta y^2 = \delta(\delta F/\delta y)/\delta y.$

THINGS TO REMEMBER :

Note 1 To find Fx , differentiate F w.r.t. x (treating other variable y as constant). In the same way to find Fy treat x as constant.

Note 2 $\delta(\vec{F}_1 + \vec{F}_2)/\delta x = \delta(\vec{F}_1)/\delta x + \delta(\vec{F}_2)/\delta x$.

Note 3 $\delta(k\vec{F}_1)/\delta x = k\delta(\vec{F}_1)/\delta x.$

Note 4 $\delta(\vec{F}_1.\vec{F}_2)/\delta x = \vec{F}_1.\delta(\vec{F}_2)/\delta x + \delta(\vec{F}_1)/\delta x.\vec{F}_2.$

Note 5 $\delta(\vec{F}_1 \times \vec{F}_2)/\delta_2 x = \vec{F}_1 \times \delta(\vec{F}_2)/\delta x + \delta(\vec{F}_1)/\delta x \times \vec{F}_2$

Note 6 $\delta(a_x i+a_y j+a_z k)/\delta t = (\delta a_x/\delta t)i + (\delta a_y/\delta t)j + (\delta a_z/\delta t)k.$

Note 7 In general vector $\vec{r}$ is taken as $\vec{r}$ =xi+yj+zk. Then |r| $= \sqrt{[x^2+y^2+z^2]}$, $|r|^2 = x^2+y^2+z^2$ and $\delta r/\delta x = x/\sqrt{(x2+y2+z2)} = x/r$ etc.

1.8.2 Partial Derivative of Vector:

Operator ∇ is generally called delta and in brief read as del and is defined by

$$\nabla = i\,\delta/\delta x + j\,\delta/\delta y + k\,\delta/\delta z$$

1.8.3 The Gradient :

The gradient of a scalar function F(x,y,z) written by grad F or ∇F is defined by

$$\text{grad } F = \nabla F = I\,\delta F/\delta x + j\,\delta F/\delta y + k\,\delta F/\delta z$$

1.8.4 The Divergence :

let F(x,y,z) = $F_1 i+F_2 j+F_3 k$ then divergence of F written as divF or ∇.F is defined by

divF ≡ ∇.F = $(\vec{i}\,\delta/\delta x + \vec{j}\,\delta/\delta y + \vec{\kappa}\,\delta/\delta z).(\vec{F}_1 \iota + \vec{F}_2 j + \vec{F}_3 \kappa) = \delta\vec{F}_1/\delta x + \delta\vec{F}_2/\delta y + \delta F_3/\delta z$

THINGS TO REMEMBER :

Note (1) Divergence of a vector F is a scalar quantity .

Note (2) A vector F is solenoidal if divF =0.

1.8.5The Curl :

let F(x,y,z) = $\vec{F}_1 i + \vec{F}_2 j + \vec{F}_3 k$ then the curl of F written as curlF or ∇xF is defined by curlF ≡ ∇xF = $(i\,\delta/\delta x + j\,\delta/\delta y + k\,\delta/\delta z) \times (F_1 i+F_2 j+F_3 k)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$=(\delta F_3/\delta y - \delta F_2/\delta z)+(\delta F_1/\delta z - \delta F_3/\delta x)+(\delta F_2/\delta x - \delta F_1/\delta y)$

THINGS TO REMEMBER :

Note (1) curl F is also known as rotation of F and written as rot F .

Note (2) curl F is a vector quantity.

Note (3) A vector F is called ir-rotational if curlF =0.

Aid to memory :

Gradient - G – general product .Divergence- D means dot product , curl – C means cross product .

SOLVED EXAMPLES

Ex.1 (a) If F $=x^3+y^3+z^3-3xyz$, find ∇F

Solv. ∇F = $(i\,\delta/\delta x + j\,\delta/\delta y + k\,\delta/\delta z)\,(x^3+y^3+z^3-3xyz,)$

= $i\,\delta\,(x^3+y^3+z^3-3xyz)/\delta x + j\,\delta\,(x^3+y^3+z^3-3xyz)/\delta y$

$+ k\,\delta\,(x^3+y^3+z^3-3xyz)/\delta z$

= $i(3x^2-3yz)+j(3y^2-3xz)+k(3z^2-3xy)$ …….Ans.

Ex.1 (b) If $\vec{r}$ = xi+yj+zk , show that div $\vec{r}$ = 3.

Soln . div $\vec{r}$ = $\nabla.\ \vec{r}$

= $(i\,\delta/\delta x + j\,\delta/\delta y + k\,\delta/\delta z).(xi+yj+zk)$

= 1+1+1 = 3 …….Ans.

Ex.1 (c) f $= x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$,then find div f and curl f at point (1,-1,1).

Soln. div f = ∇. f = (i $\delta/\delta x$+j$\delta/\delta y$+ k$\delta/\delta z$).($x^2zi-2y^3z^2j+xy^2zk$)

= $\delta(x^2z)/\delta x - \delta(2y^3z^2)/\delta y + \delta(xy^2z)/\delta z$

= $2xz-6y^2z^2+xy^2\ \vec{i}\,\delta/\delta x+\vec{j}\,\delta/\delta y+\vec{\kappa}\,\delta/\delta z.\ x^2z\vec{i}-2y^3z^2\vec{j}+xy^2z\vec{\kappa}$

∴ div f at point (1,-1,1) is

= $2.1.1-6(-1)^2.(1)^2+1.(-1)^2$

= 2-6+1 = -3 ……….Ans

now curl f = ∇x f

$=(\vec{i}\,\delta/\delta x+\vec{j}\,\delta/\delta y+\vec{\kappa}\,\delta/\delta z).(x^2z\vec{i}-2y^3z^2\vec{j}+xy^2z\vec{\kappa})$

$$= \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x^2z & -2y^3z^2 & xy^2z \end{vmatrix}$$

= i [$\delta(xy^2z)/\delta y - \delta(-2y^3z^2)/\delta z$] -

j [$\delta(xy^2z)/\delta x - \delta(x^2z)/\delta z$]+k[$\delta(-2y^3z^2)/\delta x - \delta(x^2z)/\delta y$]

= i[$2xyz+4y^3z$] –j[y^2z-x^2] +k[0-0]

=i[$2xyz+4y^3z$] –j[y^2z-x^2]

now at point (1-1 1), curl f is

= -6i+0 = -6i …………Ans.

Ex.1 (d) If $\vec{r}$ = xi+yj+zk , show that curl r = 0.

Soln. curl $\vec{r}$ = $\nabla\times\vec{r}$

= (i $\delta/\delta x$+j$\delta/\delta y$+ k$\delta/\delta z$)x(xi+yj+zk)

$$= \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x & y & z \end{vmatrix}$$

= $\sum i\,[\delta z/\delta y - \delta y/\delta z]$ = i(0)+j(0)+k(0) =0 ….**Hence proved.**

Ex.2 Show that div $(r^n.\vec{r})= (n+3) . r^n$ and curl $(r^n.\vec{r})=0$ where $\vec{r} = |r|$.

Soln. We know that $\vec{r} = xi+yj+zk$

$\therefore$ div $(r^n.\vec{r})=\nabla.(r^n.\vec{r})$

$$= \nabla.\{r^n.(xi+yj+zk)\}$$

$$= \delta(r^n.x)/\delta x+\delta(r^n.y)/\delta y+\delta(r^n.z)/\delta z \quad \text{……..(1)}$$

now

$$\delta(r^n.x)/\delta x = r^n + x.nr^{(n-1)}\,\delta r/\delta x \quad \text{[Art 1.8 , note 6]}$$

$$= r^n + x.nr^{(n-1)}\,x/r$$

$$= r^n + x^2.nr^{(n-2)}$$

similarly

$$\delta(r^n.y)/\delta y = r^n + y^2.nr^{(n-2)}$$

$$\delta(r^n.z)/\delta z = r^n + z^2.nr^{(n-2)}$$

on substituting these values in (1), we get

$$= 3\,r^n + n(x^2+y^2+z^2)\,r^{(n-2)}$$

$$=3\,r^n + n(r^2)\,r^{(n-2)} \quad \text{note (6)}$$

$$= (3+n)\,r^n$$

now curl $(r^n.\vec{r}) = \nabla\times(r^n.\vec{r}) =\nabla\times\{(r^n.xi+(r^n.yj)+(r^n.zk)$

$$= \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ r^n.x & r^n y & r^n z \end{vmatrix}$$

$$= \sum i\,[\delta(r^n.z)/\delta y - \delta(r^n y)/\delta z]$$

$$= \sum i[nr^{n-1}.z\delta r/\delta y - nr^{n-1}y\delta r/\delta z]$$

$$=\sum i[zy - yz]\,nr^{n-2}$$

$$=\sum i(0).= 0$$

Ex.3 Show that $\nabla.(\vec{r}\times\vec{a}) =0$,where a is a constant vector.

Soln. $\nabla.(\vec{r}\times\vec{a})$ =(i δ/δx+jδ/δy+ kδ/δz) . $(\vec{r}\times\vec{a})$

= i . $\delta(\vec{r}\times\vec{a})/\delta$x+ j. $\delta(\vec{r}\times\vec{a})/\delta$y+ k . $\delta(\vec{r}\times\vec{a})/\delta$z

=$\sum$ i . $\delta(\vec{r}\times\vec{a})/\delta$x

=$\sum$ i .[(δr/δx)× a + r × (δa/δx)]

= $\sum$ i . (i×a + r×0)

{since r=xi+yj+zk$\Rightarrow$ δr/δx= i & a being constant}

= $\sum$ i . (i×a)

= $\sum$ [i ia] =0 [Note 10 Art. 1.2]

Ex.4 Prove that curl curl $\vec{F}$ =0, where F =z $\vec{i}$ + x $\vec{j}$ + y $\vec{k}$.

Soln. We know that curl $\vec{\boldsymbol{F}}$ = ∇× (z $\vec{i}$ + x $\vec{j}$ +y $\vec{k}$)

$$= \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ z & x & y \end{vmatrix}$$

= i(1-0) + j(1-0)+k(1-0)

= $\vec{i}+\vec{j}+\vec{k}$ ……………..(i)

now

curl curl F = ∇× ($\vec{i}+\vec{j}+\vec{k}$)

$$= \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 1 & 1 & 1 \end{vmatrix} = 0$$ **hence proved** .

Ex.5 Show that div grad r^m = m (m+1) $r^{(m-2)}$

Soln. grad r^m =(i δ/δx+jδ/δy+ kδ/δz) r^m

=i $\delta r^m/\delta$x+j$\delta r^m/\delta$y+ k$\delta r^m/\delta$z

$= i(m r^{m-1} \, \delta r/\delta x)+j(m r^{m-1} \delta r/\delta y)+ k(m r^{m-1} \delta r/\delta z)$

$= m r^{m-1} [i(x/r)+j(y/r)+k(z/r)]$

$= m r^{m-2}(xi+yj+zk)$ ………(i)

now

div grad $r^m = \nabla$. grad r^m

$= (i \, \delta/\delta x+j\delta/\delta y+ k\delta/\delta z) . [m r^{m-2}(xi+yj+zk)]$

$= \sum m \, \delta \, (r^{m-2} x)/ \delta x$

$= \sum m [r^{m-2} +x.(m-2)r^{m-3}. \, \delta r/\delta x]$

$=\sum m [r^{m-2} +x^2.(m-2)r^{m-4}.]$

$= m [3 r^{m-2} +(x^2 +y^2 +z^2).(m-2)r^{m-4}]$

$= m [3r^{m-2} + r^2 (m-2)r^{m-4}]$

$= m[3r^{m-2}+(m-2)r^{m-2}] = m[3+m-2]r^{m-2}$

$= m(m+1)r^{m-2}$ ……………...proved

Important Note: The operator $\nabla^2 = \nabla.\nabla$ is called the Laplacian operator and defined as

$$\nabla^2 = \nabla.\nabla = \delta^2/\delta x^2+\delta^2/\delta y^2+\delta^2/\delta z^2$$

Ex.6 Show that $\nabla^2 f(r)= (2/r) f'(r)+f''(r)$.

Soln. $\nabla^2 f(r)= \nabla.\nabla \{f(r)\}$

$= \{ \delta^2/\delta x^2+\delta^2/\delta y^2+ \delta^2/\delta z^2 \} f(r)$

$= \sum \, \delta^2 [f(r)]/\delta x^2$

now

$(\delta^2/\delta x^2)[f(r)]= \delta \, [\delta f(r)/ \delta x] /\delta x$

$= (\delta/\delta x) [f '(r). \, \delta r/\delta x]$

$= (\delta/\delta x) [f '(r) (x/r)]$

$= \{ f '(r) (x/r) \, \delta r/\delta x + f '(r) [(1/r)-(x/r^2) \, \delta r/\delta x]\}$

$= \{ f''(r) (x^2/r^2) + f'(r) [(1/r)-(x^2/r^3)]\}$

$= \{ f''(r) (x^2/r^2) + f'(r)/r- f'(r)(x^2/r^3) \}$ …………(ii)

on combining (i) &(ii) , we get

$\nabla^2 f(r)=\sum \{ f''(r) (x^2/r^2) + f'(r)/r- f'(r)(x^2/r^3) \}$

$= f''(r)/r^2. (x^2+y^2+z^2) +3 f'(r)/r - f'(r)/r^3 .(x^2+y^2+z^2)$

$= \frac{f''r}{r^2} r^2 . r^2+3 f'(r)/r - \frac{f'r}{r^3} r^2$

$= f''(r) + 3[f'(r)/r] - f'(r)/r$

$= f''(r) + 2[f'(r)/r]$ ……………….Hence proved.

CHECK YOUR PROGRESS :

Q.1 If $f= x^3-y^3+xz^2$, find grad f at (1,-1,2).

Q.2 If $\vec{r} = xi+yj+zk$ then find $\nabla.r^n$.

Q.3 Find $\nabla\times F$ where $F = y(x+z)\vec{i} + z(x+y)\vec{j} + x(z+y)\vec{k}$.

Q.4 Prove that curl grad $\phi =0$.

Q.5 If $\vec{a}$ is a constant vector then show that curl $(\vec{r} \times \vec{a}) = -2\vec{a}$.

Q.6 Evaluate curl grad r^n . [Ans. 0]

Q.7 Find the condition for which the function $\phi = ax^2+by^2-cz^2$ is satisfy $\nabla^2 \phi =0$.

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UNIT-2

UNIT-2

BLOCK INTRODUCTION

Integration is a reverse process of differentiation but not exactly. It is also an important concept in Mathematics as well as in the other field of studies. With the help of integration we can also find the area, surface area and volume of the given closed curve.

The main features of this block are :

- Integration of vectors under different conditions.
- Integration along a line (curve), around closed (regular) curve.
- Surface integration and volume integration.
- Gauss, Greens & stokes theorems which are more useful in Physics.

UNIT - 2

VECTOR INTEGRATION

STRUCTURE

2.1 INTEGATION OF VECTORS

Let $\vec{a}(t)$ be a vector function and there exits a function $\vec{f}(t)$ such that $\frac{d\vec{f}}{dt} = \vec{a}(t)$ then $\vec{f}(t)$ is defined as integral of $\vec{a}(t)$ and written as.

$\int \vec{a}(t)dt = f(t)$ ....(i)

If $\vec{a}(t) = \vec{a}_x i + \vec{a}_y j + \vec{a}_z k$ and $\vec{f}(t) = \vec{f}_x i + \vec{f}_y j + \vec{f}_z k$ then from the definition if $\vec{a} = \frac{d\vec{f}}{dt}$

then $\int \vec{a}(t)dt = \int \left[a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right]dt = \vec{f}(t)$

$= f_x\vec{i} + f_y\vec{j} + f_z\vec{k} \quad \left[f_x = \int a_x dt\right]$, etc. ....

(ii)

In general if $\vec{a}(t) = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ then $\int \vec{a}(t)dt = i\int a_x dt + j\int a_y dt + k\int a_z dt$.

IMPORTANT RESULTS :

(1) We know that $\frac{d}{dt}\left(\vec{a}.\vec{b}\right) = \frac{d\vec{a}}{dt}.\vec{b} + \vec{a}.\frac{d\vec{b}}{dt}$

$$\therefore \int \left[\frac{d\vec{a}}{dt}.\vec{b} + \vec{a}.\frac{d\vec{b}}{dt}\right]dt = \vec{a}.\vec{b} + c$$

where c is a scalar quantity as the integrand is a scalar quantity.

(2) We know that

$$\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \frac{a\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$$

$$\therefore \int \left[\frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}\right]dt = \left(\vec{a} \times \vec{b}\right) + c$$

Where c is a integration constant and a vector quantity as integrand is a vector quantity.

(3) We know that

$$\frac{d}{dt}\left[(\vec{a})^2\right] = 2\vec{a}.\frac{d\vec{a}}{dt}$$

$\therefore \int\left(2\vec{a}\,\frac{d\vec{a}}{dt}\right)dt = \vec{a}^2 + c$ where c is a integration constant .

Solved examples

Ex.1 If $\vec{f}(t) = t\vec{i} + \left(t^2 - 2t\right)\vec{j} + \left(3t^2 + 3t^3\right)\vec{k}$ then find that $\int_0^1 \left[\vec{f}(t)\right]dt$

Soln. $\int_0^1 f(t)dt = \int_0^1 \left[ti + \left(t^2 - 2t\right)j + \left(3t^2 + 3t^3\right)k\right]dt$

$= \quad i\int_0^1 tdt + j\int_0^1 \left(t^2 - 2t\right)dt + k\int_0^1 \left(3t^2 + 3t^3\right)dt + c$

$= \quad i\left[\frac{t^2}{2}\right]_0^1 + j\left[\frac{t^3}{3} - t^2\right]_0^1 + k\left[t^3 + \frac{3}{4}t^4\right]_0^1 + c$

$= \quad i\left(\frac{1}{2}\right) + j\left[\frac{1}{3} - 1\right] + k\left[1 + \frac{3}{4}\right]$

$= \quad \frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$ …………..Ans.

Ex.2 If $\vec{a}(t) = 5t^2 i + \vec{t}j - t^3\vec{k}$, show that $\int_1^2 \left(\vec{a} \times \frac{d^2\vec{a}}{dt}\right)dt = -14\vec{i} + 75\vec{j} - 15\vec{k}$

Soln. Given $\vec{a} = 5t^2\vec{i} + t\,\vec{j} - t^3\vec{k}$

$\therefore \quad \frac{d\vec{a}}{dt} = 10t\ \vec{i} + \vec{j} - 3t^2\vec{k}$

and $\quad \frac{d^2\vec{a}}{dt^2} = 10\vec{i} - 6t\vec{k}$

Now $\vec{a} \times \frac{d^2\vec{a}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix}$

$= \quad \vec{i}\left[-6t^2 - 0\right] - \vec{j}\left[-30t^3 + 10t^3\right] + \vec{k}\left[0 - 10t\right]$

$= \quad -6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k}$

$\therefore \quad \int_1^2 \left(\vec{a} \times \frac{d^2a}{dt^2} \right) dt = \int_1^2 \left[-6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k} \right] dt$

$= \quad \left[-6\frac{t^3}{3}\vec{i} \right]_1^2 + \left[20\frac{t^4}{4}\vec{j} \right]_1^2 + \left[-10\frac{t^2}{2}\vec{k} \right]_1^2$

$= \quad \left[-2t^3 \right]_1^2 \vec{i} + \left[5t^4 \right]_1^2 \vec{j} - \left[5t^2 \right]_1^2 \vec{k}$

$= \quad -14\vec{i} + 75\vec{j} - 15\vec{k}$ **Proved.**

Ex.3 Evaluate $\int_1^2 (e^t\vec{i} + t^2\vec{j} - e^{-2t}\vec{k})dt$

Soln. $\int_1^2 \left(e^t\vec{i} + t^2\vec{j} - e^{-2t}\vec{k} \right) dt$

$= \quad \vec{i}\int_1^2 e^t dt + \vec{j}\int_1^2 t^2 dt - \vec{k}\int_1^2 e^{-2t} dt$

$= \quad \vec{i}\left[e^t \right]_1^2 + \vec{j}\left[\frac{t^3}{3} \right]_1^2 - \vec{k}\left[\frac{e^{-4}}{-2} \right]_1^2$

$= \quad \vec{i}\left[e^2 - e \right] + \vec{j}\left[\frac{8}{3} - \frac{1}{3} \right] + \vec{k}\left[\frac{1}{2}\left(e^{-4} - e^{-2} \right) \right]$

$= \quad \left(e^2 - e \right)\vec{i} + \frac{7}{3}\vec{j} + \frac{1}{2}\left(e^{-4} - e^{-2} \right)\vec{k}$ ………Ans.

Ex.4 Given

$\vec{a}(t) =$
- $2\vec{i} - \vec{j} + 2\vec{k}$, when $t = 2$
- $4\vec{i} - 2\vec{j} + 3\vec{k}$, when $t = 3$

Then show that $\int_2^3 \left(a.\frac{d\vec{a}}{dt} \right) dt = 10$

Soln. We know that

$$\frac{d}{dt}\left[\frac{1}{2}\vec{r}^2 \right] = \vec{r}\,\frac{d\vec{r}}{dt}$$

$\therefore \quad \frac{1}{2}r^2 = \int\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt$ …………(i)

Now as per question and (i) we have

$\int_2^3\left(\vec{a}\frac{d\vec{a}}{dt}\right)dt = \left[\frac{1}{2}a^2\right]_2^3$ ....(ii)

$= \quad \frac{1}{2}\left[\left\{|\vec{a}|^2\right\}_{t=3} - \left\{|\vec{a}|^2\right\}_{t=2}\right]$

$\therefore$ square or vector = square of its modulus

$= \quad \frac{1}{2}\left\{\left[(4)^2 + (-2)^2 + (3)^2\right] - \left[(2)^2 + (-1)^2 + (2)^2\right]\right\}$

$= \quad \frac{1}{2}\{(16+4+9)-(4+1+4)\}$

$= \quad \frac{1}{2}[29-9]$

$= \quad 10$ Proved

CHECK YOUR PROGRESS :

Q.1 Evaluate - $\int_0^1\left(e^t i + e^{-2t} j + tk\right)dt$

Ans. $\left[(e-1)j - \frac{1}{2}\left(e^{-2}-1\right)j + \frac{1}{2}k`\right]$

Q.2 Evaluate - $\int_1^2 \vec{r}.\left(\frac{d^2\vec{r}}{dt^2}\right)dt$ where $\vec{r}(t) = 2t^2 i + tj - 3t^3 k$

Ans. $[-42i + 90j - 6k]$

Q.3 Find $\vec{r}$ satisfying the equation.

$\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$, $\vec{a}$ and $\vec{b}$ are constant vectors.

Ans. $\left[\frac{1}{6}t^3\vec{a} + \frac{1}{2}t^2\vec{b} + t.c + d\right]$

2.1 Line, Surface and Volume Integral

2.2.1 Definition of Line Integral :

Let F (r) be a continuous vector function defined on a smooth curve C given by $r = f(t)$. If S denote the actual length of any point $P(x, y, z)$ from a fixed point on the curve, then $\frac{dr}{ds}$ is a unit vector along the tangent to the curve at P.Then

$$\int_c \left(F.\frac{dr}{ds} \right) ds \quad or \quad \int_c F.\, dr. \qquad \text{..................(1)}$$

is called the tangent line integral of F along curve C.

THINGS TO REMEMBER:

Note 1 : Relation (1) may also be written as $\int_c F\left(\frac{dr}{dt} \right) dt.$

Note 2 : If the path of integration C is piecewise curve and joined end to end by finite number of curves $C_1, C_2, C_3 .... C_n$ then (1) written as

$$\int_c F.dr = \int_{c1} F.dr + \int_{c2} F.dr + \int_{c3} F.dr + ... + \int_{cn} F.dr$$

Note 3: If the given curve is a closed curve then the line integral is symbolically written as $\oint$ *instead* of $\int$.

Note 4 : We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $dr = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$

Solved Examples

Ex.1 Evaluate $\int_c F.dr$ where $F = x^2\vec{i} + y^3\vec{j}$ and the curve C is the are of the parabola $y = x^2$ in xy – plane from $(0,0)$ to $(1,1)$.

Soln. Given curve is $y = x^2 \Rightarrow dy = 2xdx$ .....(1)

$\therefore$ F = $x^2\vec{i} + y^3\vec{j} = x^2\vec{i} + (x^2)^3\vec{j}$

and

$dr \equiv \vec{i}\,dx + \vec{j}dy = \vec{i}\,dx + \vec{j}(2xdx)$ by (i)

$\therefore$ Required line integral

$$\int_c F.dr \equiv \int_{x=0}^{x=1} F.dr = \int_{x=0}^{x=1} (x^2 i + x^6 j).(idx + 2\times jdx)$$

$$= \int_0^1 (x^2 dx + 2x^7 dx)$$

$$= \left[\frac{x^3}{3} + \frac{2}{8}x^3\right]_0^1$$

$$= \frac{1}{3} + \frac{2}{8}$$

$$= \frac{8+6}{24} = \frac{14}{24} = \frac{7}{12}$$ Ans.

Alternate Method: If we put x = t in the given equation of curve $y = x^2$ we get $y = t^2$.

$\therefore$ $F \equiv x^2\vec{i} + y^3\vec{j} = t^2\vec{i} + (t^2)^3\vec{j}$

$= t^2\vec{i} + t^6\vec{j}$

and $dr = idx + jdy = idt + j(2tdt)$

[since $x = t \Rightarrow dx = dt$ and $y = t^2 \Rightarrow dy = 2tdt$]

$\therefore$ Required line integers

$$\int_c F.dr = \int_{t=0}^{t=1} [t^2 i + t^6 j].[idt + 2tjdt]$$

$$= \left[\frac{t^3}{3} + \frac{1}{4}t^8\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{1}{4}\right] = \frac{7}{12}$$ Ans.

Ex. 2: If $F = (2x + y)\vec{i} + (3y - x)\vec{j}$ then evaluate $\int_c F.dr$ where C is the curve in *xy*-plane consisting the straight line from (0,0) to (2,0) and then to (3,2)

Sol. In the xy-plane

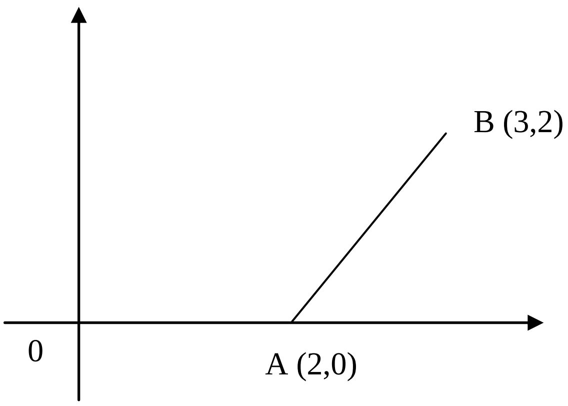
$\vec{r} = x\vec{i} + y\vec{j}$

$\therefore \; d\vec{r} = dx\vec{i} + dy\vec{j} + z\vec{k}$

$\therefore$ F.dr = (2x + y)dx + (3y - x)dy .....(i)

Now the equation of a straight line passing through two given point A(2,0) and B(3,2) is

$$(y-0) = \frac{2-0}{3-2}(x-2) \Rightarrow y = 2x - 4$$



$\therefore$ Path of integration C consist line OA and AB as shown in the figure.

$\therefore$ Required line integral

$\int_c F.Dr = \int_{C_1=OA} F.dr + \int_{C_2=AB} F.dr$ .....(ii)

Now $\int_{C_1} (F.dr) = \int_{OA} (2x + y)dx + (3y - x)dy$

$= \int_{x=0}^{x=2} 2x dx$

[$\therefore$ Since on DA $y = 0 \Rightarrow dy = 0 \,\&\, x$ *varies from* 0 to 2]

$= \left[x^2\right]_{x=0}^{x=2} = 4$ .....(iii)

Now $\int_{C_2} F.dr = \int_{AB} (2x + y)dx + (3y - x)dy$

$$= \int_{x=2}^{x=3} [2x + 2x - 4]dx + [3(2x-4) - x2.dx$$

[Since on AB y = $2x - 4 \Rightarrow dy = 2dx$]

$$= \int_{x=2}^{x=3} (4x-4)dx + (10x - 2y)dx$$

$$= \int_{x=2}^{x=3} (14x - 280\,dx$$

$$= \left[7(x^2) - 28x\right]_{x=2}^{x=3}$$

= 7(9–4)–28(3–2)

= 35 – 28 = 7 .....(iv)

From (ii) (iii) and (iv)

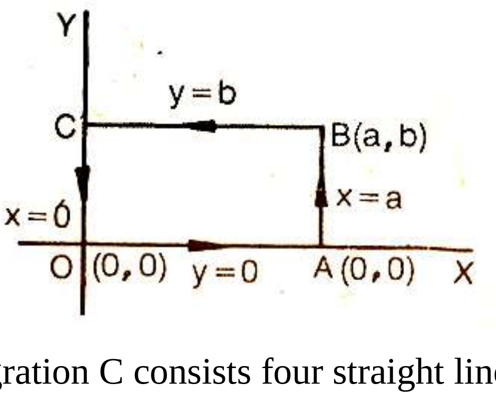
$\int_c F.dr = 4 + 7 = 11$ Ans.

Ex.3 Evaluate $\int_c F.dr$ where $F = (x^2 + y^2)i - 2xyj$ curve C is the rectangle in xy-plane bounded by y = 0, x = a, y = b, x = 0

Sol. In the xy plane we have

$r = xi + yj \Rightarrow dr = idx + jdy$

$\therefore$ F.dr = $(x^2 + y^3dx - 2xydy)$ .....(i)



Clearly the path of integration C consists four straight line OA, AB, BC and CO as shown in the diagram.

$\therefore$ Required line integral

$\int_c F.dr = \int_{OA} F.dr + \int_{AB} F.dr + \int_{BC} Fdr + \int_{CO} Fdr$ .....(ii)

Clearly on OA $y = 0 \Rightarrow dy = 0$ and x varies from 0 to a.

On AB $x = a \Rightarrow dx = 0$ and y varies from 0 to b.

On BC $y = b \Rightarrow dy = 0$ and x varies from a to 0

and on CO $x = 0 \Rightarrow dx = 0$ and y varies from b to 0

$\therefore \int_{OA} F.dr = \int_{OA} (x^2 + y^2)dx - 2xydy$

$$= \int_{x=0}^{x=a} x^2 dx \qquad \text{as y = 0 \& dy = 0}$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad \text{....(iii)}$$

$$\int_{AB} F.dr = \int_{y=0}^{y=b} -2aydy \qquad [\because \text{ x = a \& dx = 0}]$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2 \qquad \text{....(iv)}$$

$$\int_{BC} F.dr = \int_{x=a}^{x=0} (x^2 + b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0 \qquad [\because \text{ y = b \& dy = 0}]$$

$$= \frac{-a^3}{3} - ab^2$$

and $\int_{Co} F.dr = \int_{y=a}^{y=0} 0\, dy = 0$ .....(vi)

On putting the values from (iii) (iv) (v) and (vi) in (ii), we get required result i.e.

$$\int_c F.dr = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2 \qquad \text{Ans.}$$

Ex.4 Evaluation $\int_c F.dr$ where $F = xyi + yzj + zxk$ and C is $r = ti + t^2 j + t^3 k$, t varying from –1 to +1.

Soln. We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore$ Parametric equation of given curve C is

$x = t, y = t^2, z = t^3$ and so we have

$$F = xyi + yzj + zxk$$

$$= (t)(t^2)i + (t^2)(t^3)j + t^3.tk = t^3 1^0 + t^5 j + t^4 k \qquad .....(i)$$

and $\frac{dr}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$ .....(ii)

$\because r = ti + t^2 j + t^3 k$]

Now required line integral

$$\int_c F.dr = \int_{t=-1}^{t=1} (F.\frac{dr}{dt})dt$$

$$= \int_{t=-1}^{t=1} (t^3 + 2t^6 + 3t^6)dt \qquad \text{[by (i) and (ii)]}$$

$$= \int_{-1}^{1} (t^2 + 5t^6)dt$$

$$= \left[\frac{t^4}{4} + \frac{5}{7}t^7\right]_{-1}^{1}$$

$$= \left(\frac{1}{4} + \frac{5}{7}\right) - \left(\frac{1}{4} - \frac{5}{7}\right)$$

$$= \frac{10}{7} \qquad \text{Ans.}$$

Ex. 5: Evaluate $\int_c F.dr$ where

F = $(3x^2 + 6y)i - 14yzj + 20xz^2 k$

along the curve $x = t, y = t^2, z = t^3$ from point (0,0,0) to (1,1,1)

Sol. Here r = $ti + t^2 j + t^3 k$

$\therefore \frac{dr}{dt} = i + 2tj + 3t^2 k$ ....(i)

also

F = $(3t^2 + 6t^2)i - 14t^5 j + 20t^7 k$

= $9t^2 i - 14t^5 j + 20t^7 k$ ....(ii)

$\therefore$ F.dr = $(9t^2 - 28t^6 + 60t^9)$

So the required line integral is

$$\int_c F.dr = \int_C (F.\frac{dr}{dt})dt$$

$$= \int_{t=0}^{t=1} (9t^2 - 28t^6 + 60t^9)dt$$

$$= \left[3t^2 - 4t^7 + 6t^{10}\right]_0^1$$

= 3 – 4 + 6 = 5 Ans.

2.2.2 Surface integral

Let F(r) be a continuous vector point function and r = f(u,v) be a smooth surface S. Let S be a two sided surface in which one side is treating as positive side (outer side in case of a closed surface). Then the surface integral of F(r) over S is denoted by

$$\int_S F.nds \text{ or } \iint_S F.dS \quad .....(1)$$

where dS = nds and n denotes the unit vector normal to the surface.

THINGS TO REMEMBER:

NOTE: 1. Cartesian formula for the surface integral (1) is given by

$$\iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Where F = $F_1(xyz)i + F_2(xyz)j + F_3(xyz)k$

NOTE: 2. The surface integral over a closed surface S is denoted by $\oint s$

NOTE: 3. In xy plane ds = $\dfrac{dxdy}{|n.k.|}$

yz plane ds = $\dfrac{dydz}{|n.i|}$

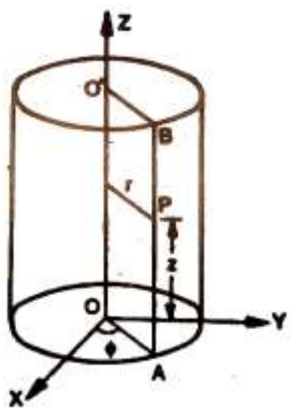
and inzx plane dx = $\dfrac{dxdz}{|n.j|}$ (conditional)

NOTE: 4 Unit vector normal to the given surface ϕ is given by

$$\hat{n} = \frac{grad\phi}{|grad|} = \frac{\nabla\phi}{|\nabla\phi|}$$

Solved Example

Ex.1 Evaluate $\iint_S F.nds$ where F = $zi + xj - 3y^2zk$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.



Soln. Here the projection of the given surface S on xz plane as the region OABC bounded by x-axis (BC || x axis) and z-axis (AB || z axis) as shown in the figure.

∴ Required surface integral

$$\iint_S F.\hat{n}ds = \iint_R F.\hat{n}.\frac{dxdz}{|n.j|} \quad ...(1)$$

[Note (3)] (region R = OABC)

Now $\hat{n} = \dfrac{grad\phi}{|grad\phi|}$ [Note (4) $\phi = x^2 + y^2 = 16 = 0$

$$= \frac{1}{4}(xi + yj) \qquad ...(2)$$

Since grad ϕ = $\nabla\phi$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 - 16)$$

$$= 2xi + 2yj$$

and |gradϕ| = $\sqrt{4x^2 + 4y^2}$

$$= \sqrt{4(x^2 + y^2)}$$

$$= \sqrt{4(16)}$$

$$= \sqrt{64} = 8$$

$\therefore$ $\boldsymbol{\hat{n}.j}$ = $\frac{1}{4}(xi + yj).j$

$$= \frac{1}{4}y$$

From (i) = $\iint_R F.n\frac{dxdz}{|n.j|}$

$$= \iint_R (zi + xj - 3y^2 zk).\left(\frac{1}{4}(xi + yj)\right).\frac{dxdz}{(1/4, y)}$$

$$= \iint_R \left(\frac{3x + yx}{y}\right)dxdz$$

$$= \iint_R \left[x + \frac{3x}{y}\right]dxz$$

$$= \int_{x=0}^{4}\int_{z=0}^{5} x + \frac{3x}{\sqrt{16 - x^2}}\,dxdz$$

$$= \int_{x=0}^{4}\left[\int_{0}^{5}\left\{x + \frac{3x}{\sqrt{16 - x^2}}\right\}dz\right]dx$$

$$= \int_{x=0}^{4}\left[xz + \frac{x}{2\sqrt{16 - x^2}}z^2\right]_0^5 dx$$

$$= \int_{x=0}^{4}\left[5x+\frac{x}{2\sqrt{16-x^2}}(25)\right]_0^5 dx$$

$$= \frac{5}{2}\left[x^2\right]_0^4+\frac{25}{2}\left[-\frac{1}{2}\int_0^4\frac{-2x}{\sqrt{16-x^2}}dx\right]$$

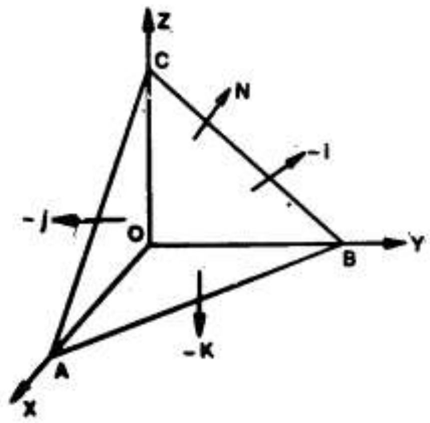
$$= \frac{80}{2}+\frac{25}{2}\left(-\frac{1}{2}\right)\left[\frac{\sqrt{16-x^2}}{(1/2)`}\right]_0^4$$

$$= 40-\frac{25}{2}(0-4)$$

$$= 40+50 = 90 \qquad \text{Ans.}$$

Ex.2 Evaluate $\iint_S F.nds$ where $F = x^2 i + y^2 j + z^2 k$ and S is that portion of the plane x + y + z = 1 which lies in the first octant.

Soln. According to the question the required Region R is bounded by x axis, y-axis and straight line x+ y = 1 as shown in the diagram.



$\therefore$ surface integral

$$\iint_S F.nds = \iint_R F.n.\frac{dxdy}{|n.k|} \qquad ...(1)$$

Now

$$\hat{n} = \frac{grad\phi}{|grad\phi|} = \frac{\nabla\phi}{|\nabla\phi|} \text{ where } \phi = x+y+z-1=0$$

Here $\nabla\phi$ = $\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x+y+z-1)$

= i + j + k

and $|\nabla\phi|$ = $\sqrt{3}$

$\therefore\ \hat{n}$ = $\frac{1}{\sqrt{3}}(i+j+k)$ and $\hat{n}$.k. = $\frac{1}{\sqrt{3}}$

$\therefore$ by (1)

$$\iint_R F.\hat{n}.\frac{dxdy}{|n.k|} = \iint_R\left(x^2i+y^2j+z^2k).\frac{1}{\sqrt{3}}(i+j+k)\frac{dxdy}{(1/\sqrt{3})}\right)$$

$$= \iint_R (x^2+y^2+z^2)dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} x^2+y^2+(1-x-y)^2\,dxdy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} x^2+y^2+(1-x-y)^2 dy\right]dx$$

$$= \int_0^1\left[x^2y+\frac{y^3}{3}-\frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}+\frac{(1-x)^3}{3}\right]dx$$

$$= \int_0^1\left[x^2-x^3+\frac{2(1-x)^3}{3}\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{2}{3}\frac{(1-x)^4}{4(-1)}\right]_0^1$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{6}=\frac{4-3+2}{12}=\frac{1}{4}$$ Ans.

Ex.3 Evaluate $\iint_S (y^2z^2i+z^2x^2j+x^2y^2k).ds$ where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane.

Sol. According to the question it is clear that the region is bounded by the circle $x^2 + y^2 = 1, z = 0$ in xy-plane.

Here F $= y^2z^2i + z^2x^2j + x^2y^2k$

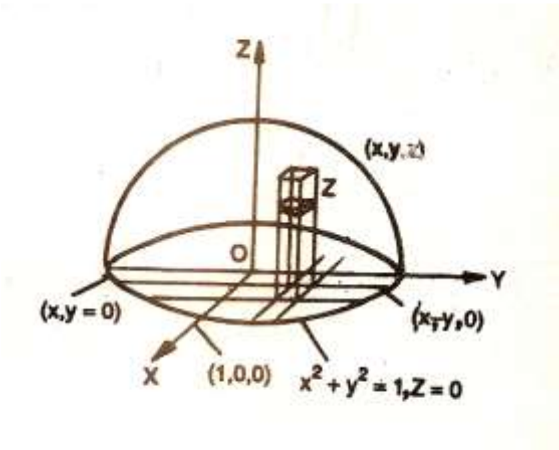
$\phi = x^2 + y^2 + z^2 = 1$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= 2xi + 2yj + 2zk$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2 \qquad [\because x^2 + y^2 + z^2 = 1]$$

$$\therefore\ \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi + yj + zk)}{2} = (xi + yj + zk)$$



Now the required surface integral is

$$\iint F.\hat{n}.\frac{dxdy}{|n.k.|} = \iint_R (xy^2z^2 + yz^2x^2 + zx^2y^2)\frac{dxdy}{z}$$

$$= \iint_R \left[(xy^2 + x^2y)z^2 + zx^2y^2\right]\frac{dxdy}{z}$$

$$= \iint_R \left[(xy^2 + x^2y)z + x^2y^2\right]dxdy$$

$$= \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \left[(xy^2 + x^2y)z + x^2y^2\right]dxdy$$

$$= \int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \left[(xy^2 + x^2y)\sqrt{1 - x^2 - y^2} + x^2y^2\right]dxdy$$

$$= \quad 2\times 2\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 dxdy$$

$$\left[\because \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, f(x) \, is\, even \\ 0, f(x) is \ odd \end{cases}\right]$$

$$= \quad 4\int_0^1 x^2 \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \quad \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2} dx$$

put x = sinθ dx = cosθdθ

and limits are θ = 0 to θ = π/2

$$= \quad \frac{4}{3}\int_0^{\pi/2} \sin^2\theta(1-\cos^2\theta)^{3/2} \cos\theta d\theta$$

$$= \quad \frac{4}{3}\int_0^{\pi/2} \sin^2\theta \cos^4\theta d\theta$$

$$= \quad \frac{4}{3}\frac{\left|\frac{2+1}{2}\right. . \left|\frac{4+1}{2}\right.}{2\left|\frac{2+4+2}{2}\right.}$$

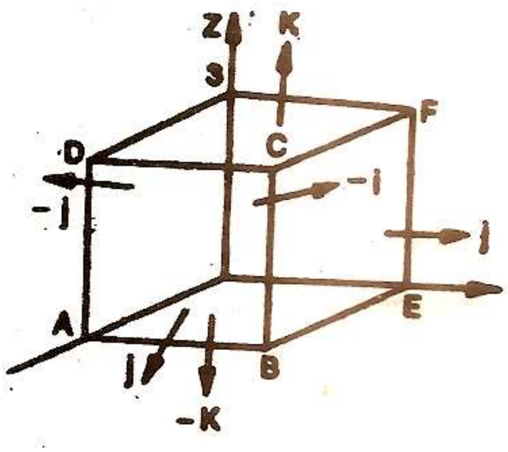
$$= \quad \frac{4}{3}\left\{\frac{\left|3/2\right.\left|5/2\right.}{2\left|4\right.}\right\}$$

$$= \quad \frac{2}{3}\frac{\left(\frac{1}{2}\sqrt{\pi}\right)\left(\frac{3}{2}\frac{1}{2}\sqrt{\pi}\right)}{(3.2.1)}$$

$$= \quad \frac{\pi}{24}$$ Ans.

Ex.4 Evaluate $\iint_S F.nds$ where F = $2yxi - yzj + x^2k$ over the surface S of the cube bounded by the corrdinate planes and planes x = a, y = a, z = a.

Sol. Clearly the surfaces of the cubes are DEFA, CDBO, EGBF, DCOA, DEGC and OAFB namely S_1,S_2,S_3,S_4, S_5 and S_6 respectively. Then we have required surface integral



$$\iint_S F.nds = \iint_{S_1} F.nds + \iint_{S_2} F.nds + \iint_{S_3} F.nds + \iint_{S_4} F.nds + \iint_{S_5} F.nds + \iint_{S_6} F.nds \qquad ....(1)$$

Face S_1 is in yz plane, therefore $\hat{n} = i, x = a$

$$\therefore \iint_{S_1} F.nds = \int_{y=0}^{a}\int_{z=0}^{a}\left(2yxi - yzj + x^2k\right).i\, dydz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a}(2ya)\, dydz \qquad [\because x = a]$$

$$= \int_{0}^{a} 2ay[z]_0^a\, dy$$

$$= 2a^2\left[\frac{y^2}{2}\right]_0^a = a^4 \qquad ....(2)$$

For the face S_2 i.e. CGBO x = 0, $\hat{n} = -i$

$$\therefore \iint_{S_2} F.nds = \int_{y=0}^{a}\int_{z=0}^{a}(-2yx)\, dydz = 0 \qquad [\because x = 0] \qquad ....(3)$$

For the face S_3 i.e. EGBF y = a, $\hat{n} = j$

$$\iint_{S_3} F.nds = \int_0^a\int_0^a(-yz)dxdz$$

$$= \int_0^a\int_0^a (-az)dxdz \qquad [\because y = a]$$

$$= -a\int_0^a \left[\frac{z^2}{2}\right]_0^a dx$$

$$= -\frac{a3}{2}[x]_0^a = -\frac{a^4}{2} \qquad ....(4)$$

For the face S_4 DCOA y = 0, $\hat{n} = -j$

$$\iint_{S_4} F.nds = \int_0^a\int_0^a (-yz)dxdz = 0 \qquad [\because y = 0] \qquad ....(5)$$

For the face S_5 DEGC z = a, $\hat{n} = k$

$$\iint_{S_5} F.nds = \int_0^a\int_0^a (x^2)dxdy \qquad [\because z = a]$$

$$= \int_0^a x^2 [y]_0^a dx$$

$$= a\int_0^a x^2 dx = a\left[\frac{a^3}{3}\right] = \frac{a^4}{3} \qquad ....(6)$$

For the face S_6 OAFD z = 0, $\hat{n} = -k$

$$\iint_{S_6} F.nds = -\int_0^a\int_0^a (x^2)dxdy = -\frac{a^4}{3} \qquad ....(7)$$

By using 2,3,4,5,6,7, (1) reduce to

$$\iint_S F.nds = \frac{1}{2}a^4 \qquad \text{Ans.}$$

2.2.3 Volume Integral:

Let F(r) be a continuous vector function and let volume V is enclosed by a smooth surface S. Then the volume integral of F(r) over V is denoted by

$$\int_V F(r)dr = \int_F Fdv = \iiint_V Fdv$$

Where dv = dxdydz (In triple integral)

THINGS TO REMEMBER:

NOTE (1): If F = $F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ then volume integral

$$\int_V F dv = i\iiint F_1 dxdydz + j\iiint F_2 dxdydz + k\iiint F_3 dxdydz$$

NOTE (2) : If ϕ is a scalar function, then volume integral is given by $\int_V \phi\, dv.$

Solved Example

Ex.1 Evaluate $\int_V F dv$ where $F = 2xzi - xj + y^2k$ and V is the region bounded by the surfaces x=0, y = 0, z = 0, x = 1, y = 1 and z = 1.

Sol. Required volume integral =

$$= \int_V F dv = \iiint F dxdydz$$

$$= \int_0^1\int_0^1\int_0^1 (2xzi - xj + y^2k) dxdydz$$

$$= \vec{i}\int_0^1\int_0^1\int_0^1 2xz dxdydz - \vec{j}\int_0^1\int_0^1\int_0^1 x dxdydz + \vec{k}\int_0^1\int_0^1\int_0^1 y^2 dxdydz$$

$$= \vec{i}.2\left[\frac{x^2}{2}\right]_0^1 [y]_0^1 \left[\frac{z^2}{2}\right]_0^1 - \vec{j}\left[\frac{x^2}{2}\right]_0^1 [y]_0^1 [z]_0^1 + \vec{k}\left[\frac{y^3}{3}\right]_0^1 [x]_0^1 [z]_0^1$$

$$= \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{3}\vec{k}$$ .....Ans.

Ex.2 Evaluate $\int_V F dv$ where $F = xi + yj + zk$ and V is the region bounded by the surfaces x=0, y = 0, y = 6, z = x^2 and z = 4.

Sol. Required volume integral is

$$= \int_V F dv = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} F dxdydz$$ [As z = x^2, z = 4 we have x = 2]

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}(xi+yj+zk)dxdydz$$

$$= i\int_0^2\int_0^6\int_{x^2}^4 xdxdydz - j\int_0^2\int_0^6\int_{x^2}^4 ydxdydz + k\int_0^2\int_0^6\int_{x^2}^4 zdxdydz$$

$$= i\int_0^2\int_0^6 x[z]_{x^2}^4\,dxdy + j\int_0^2\int_0^6 y[z]_{x^2}^4\,dxdy + k\int_0^2\int_0^6 [\frac{z^2}{2}]_{x^2}^4\,dxdy$$

$$= i\int_0^2 x(4-x^2)[y]_0^6\,dx + j\int_0^2 (4-x^2)\left[\frac{y^2}{2}\right]_0^6 dx + \frac{k}{2}\int_0^2 (4^2-x^4)[y]_0^6\,dx$$

$$= 6i\int_0^2 (4x-x^3)dx + 18j\int_0^2 (4-x^2)dx + 3k\int_0^2 (16-x^4)dx$$

$$= 6i\left[4\left\{\frac{x^2}{2}\right\}_0^2 - \left\{\frac{x^4}{4}\right\}_0^2\right] + 18j\left[(4x)_0^2 - \left\{\frac{x^3}{3}\right\}_0^2\right] + 3k\left[\{16x\}_0^2 - \left\{\frac{x^5}{5}\right\}_0^2\right]$$

$$= 6i[8-4] + 18j\left[8-\frac{8}{3}\right] + 3k\left[32-\frac{32}{5}\right]$$

$$= 24\vec{i} + 96\vec{j} + \frac{384}{5}\vec{k}$$ Ans.

CHECK YOUR PROGRESS :

Q.1 Evaluate $\int_c F.dr$ where F = $x^2y^2\vec{i} + y\vec{j}$ and c is $y^2 = 4x$ in the xy-plane from (0,0) to (4,4). Ans.[264]

Q. 2 Evaluate $\int_c F.dr$ where F = $y\vec{i} - x\vec{j}$ and c is arc of the parabola $y = x^2$ from (0,0) to (1,1) Ans [-1/3]

Q. 3 Evaluate $\int_c F.dr$, where F = $(3x^2+6y)i - 14yzj + 20xz^2k$ and curve c is x = t, y = t^2 from (0,0,0) to (1,1,1). Ans. [5]

Q. 4 Evaluate $\int_c F.dr$, where $F = xy\vec{i} - (x^2+y^2)\vec{j}$ and c is rectangle in the xy-plane bounded by the lines y =2, x=4, y=10 and x = 1 Ans. [60]

Q. 5 Evaluate $\iint_S F.n.ds$ where $F = 2x^2 i - y^2 j + 4xzk$ and the region is in the first octant bounded by $y^2 + z^2 = 9, x = 0\ and\ x = 2.$ Ans. [36(π-1)]

Q. 6 Evaluate $\iint_S F.n.ds$ where $F = 6zi - 4xj + yk$ and S is the part of plane $2x + 3y + 6z = 12$ in the first octant. Ans. [-16]

Q. 7 Evaluate $\iint_S F.n.ds$ where $F = (3/8)xyz$ and S is the surface of the cylinder $x^2 + y^2 = 16$ in the first octant between z = 0 and z = 5.

Ans.[100(i+j)]

Q. 8 Evaluate $\iint\int_V Fdv$ where F = $45x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0, z = 0.$ Ans. [128]

Q. 9 Evaluate $\int_v rdv$ where $= xi + yj + zk$ and V is the region is bounded by x =0, y=0, y = 6, z= x^2 and z = 4. Ans. [24(i+4j+(16/3)k)]

2.3 Gauss Divergence Theorem:

Statement - If V is the volume bounded by a closed surface S and F is a vector point function with continuous derivatives, then

$$\int_S F.n.ds = \int_v divFdv, \qquad .....(i)$$

where n is the unit vector outward drawn normal vector to the surface S.

Note(1) With the help of this theorem we can express volume integral as surface integral or surface integral as volume integral.

Note(2): If the given surface is of a standard curve, then we can directly obtain volume by applying volume formula of corresponding curve.

Working Rule:

Step 1 - Find div. F i.e. $\nabla.F$

Step 2 - Find integration limit using the condition given in the question.

Step 3 - Solve (i) after putting the value of div. F (i.e. $\nabla.F$), integration limit and replacing dv = dxdydz.

Solved examples

Ex. 1 Evaluate $\int\int_S F.dx$ where $F = 4xyi + yzj - xzk$ and S is the surface of the cube bounded by the plane x= 0, x = 2, y = 2, z = 0 and z = 2.

Sol. Given $F = 4xyi + yzj - xzk$

$$\therefore divF = \nabla.F$$

$$= \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z} \right).(4xyi + yzj - xzk)$$

$$= 4y + z - x$$

$\therefore$ By Guass div. theorem we have

$$\int_S F.n.ds = \int_V divFdv$$

$$= \iint\int_V (4y + z - x)dxdydz$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2}\int_{z=0}^{z=2}(4y + z - x)dxdydz$$

$$= \int_{x=0}^{x=2}\left[\int_{y=0}^{y=2}(8y + 2 - 2x)dy\right]dx$$

$$= \int_{x=0}^{x=2}\left[(4y^2 + 2y - 2xy\right]_{y=0}^{y=2}dx$$

$$= \int_{x=0}^{x=2}[(16 + 4 - 4x)]dx$$

$$= \int_{x=0}^{x=2}[(20 - 4x)]dx$$

$$= \left[20x - 2x^2\right]_{x=0}^{2=2}dx$$

$$= 40 - 8 = 32$$ Ans.

Ex.2 Evaluate $\int_S F.n.ds$ where $F = axi + byj + czk$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Sol. Here F $= axi + byj + czk$

$\therefore$ div F $= \nabla.F$

$$= \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z} \right)(axi + byj + czk))$$

$= (a + b + c)$

By Guass div. theorem we have

$$\int_S F.n.ds = \int_v divFdv$$

$$= \iint \int_V (a+b+c)dxdydz$$

$$= (a+b+c)\int_V dv$$

$$= (a+b+c)v = (a+b+c)[4/3\pi(1)^3]$$

[V=volume of sphere (Note2)]

$$= \frac{4}{3}\pi(a+b+c)$$ Ans.

Ex. 3 Verify the Gauss divergence theorem for the function $F = yi + xj + z^2k$ over the cylindrical region S bounded by $x^2 + y^2 = a^2, z = 0$ and $z = h$

Soln. By Gauss theorem we have

$$\int_S F.n.ds = \int_v divFdv \quad .....(i)$$

L.H.S.(i) $= \int_S F.n.ds$

$$= \int_S (yi + xj + z^2k).k\ dxdy$$

$$= \int_{x=-a}^{x=a} y = \sqrt{(a^2 - x^2} \int^{y=\sqrt{(a^2-x^2}} Z^2 dxdy \quad \text{[Note]}$$

according to question at bottom z = 0 and at the top of cylinder z = h, therefore at bottom $\int_S F.n.ds = 0$

and at the top $\int_S F.n.ds = \int_{x=-a}^{x=a} y = \sqrt{(a^2 - x^2} \int^{y=\sqrt{(a^2-x^2}} h^2 dxdy$

$$= 2.2\int_0^a 0 \int^{\sqrt{(a^2-x^2}} h^2 dxdy$$

$$= 4h^2 \int_0^a \sqrt{(a^2 - x^2)} dx$$

$$= 4h^2 \left[(1/2) \times \sqrt{a^2 - x^2} + (1/2) a^2 \sin^{-1}(x/a) \right]_0^a$$

$$= 4^2 \left[0 + (1/2) a^2 (\pi/2) \right]$$

$$= \pi a^2 h^2$$

$\therefore$ the entire surface integral $\int_S F.n.ds = \pi a^2 h^2$ .....(ii)

Now R.H.S. (i) = $\int_v divF dv$,

$$= \iiint \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z} \right) . \left(yi + xj + z^2 k) dxdydz \right)$$

$$= \iiint 2z \, dxdydz$$

[here x=-a to a, y = $\sqrt{(a^2 - x^2)} to \sqrt{(a^2 - x^2)} \ \& \ z = 0 \ to \ h$]

$$= 2.2 \int_0^a \int_0^{\sqrt{(a^2-x^2}} [z^2]_0^h dxdy$$

$$= 4h^2 \int_0^a \int_0^{\sqrt{(a^2-x^2}} h^2 dxdy$$

$$= 4h^2 \int_0^a \sqrt{(a^2 - x^2)} dx$$

$= \pi a^2 h^2$ ....(iii)

[since $\int_0^a \sqrt{(a^2 - x^2) dx} = \pi a^2 / 4$

Hence theorem verified from (ii) and (iii)

CHECK YOUR PROGRESS:

By using Gauss theorem

Q.1 Evaluate $\int_S F.n.ds$ where $F = xi + yj + zk$ and S is the entire surface of the cube bounded by the co-ordinates plane and x = a, y = a, z = a. Ans. [$3a^3$]

Q.2 Evaluate $\int_S F.n.ds$ where $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - yx)k$ and s is the surface of the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c.$

Q.3 If V is the volume enclosed by a closed surface S and $F = xi + 2yj + 3zk$, show that $\int_s F.n\ ds = 6V$.

2.4 Stoke's Theorem:

Statement - If S is open two sided surface bounded by a closed curve C, then for the vector F having continuous derivatives,

$$\int_c F.dr = \int (curl\ F).n\ ds, \qquad ....(i)$$

where n is the unit vector outward drawn normal vector to the surface S and c is transversal in the positive direction.

Note: With the help of this theorem we can express a line integral as a surface integral.

Working Rule: Given vector function is *F*, surface is *S* and curve is *C*.

Step 1 - Find curl F i.e. $\nabla \times F$

Step 2 - find unit normal vector $\hat{n}$ and replace *ds* as Art. 2.2.2

Step 3 - Solve after putting above values in R.H.S (i).

For verification of the theorem: Find line integral as per Art 2.21, which is L.H.S. (i). If L.H.S. (i). = RHS (i), then the theorem is verified.

Solved examples

Ex 1 Verify Stoke's theorem for $F = (x^2 + y^2)i + 2xyj$, where C is the rectangle in xy=plane bounded by $y = 0$,$x = a$,$y = b$ and $x = 0$.

Soln. In the xy-plane we have

$$\vec{r} = x\vec{i} + y\vec{j} \qquad \therefore dr = \vec{i}dx + \vec{j}dy$$

$$\therefore F.dr = (x^2 + y^2)dx - 2xydy \qquad \text{........(i)}$$

clearly the path of integration C consists four straight line OA, AB, BC & CO as shown in fig.

$\therefore$ Required line integral

$$\int_c F.dr = \int_{OA} F.dr + \int_{AB} F.dr + \int_{BC} F.dr + \int_{CO} F.dr \qquad \text{..........(ii)}$$

clearly on (1) $OA \;\; y = 0 \Rightarrow dy = 0$ and x varies from 0 to b

(2) AB $x = a \Rightarrow dx = 0$ and y varies from 0 to b

(3) BC $y = b \Rightarrow dy = 0$ and x varies from a to 0

(4) CO $x = 0 \Rightarrow dx = 0$ and y varies from b to 0

$$\therefore \int_{OA} F.dr \int_{OA} (x^2 + y^2)dx - 2xydy = \int_{x=0}^{x=a} x^2 dx = a^3/3 \qquad \text{........(iii)}$$

$$\int_{AB} F.dr = \int_{AB} (x^2 + y^2)dx - 2xydy = \int_{y=0}^{y=b} (-2ay)by = -ab^2 \qquad \text{.......(iv)}$$

$$\int_{BC} F.dr = \int_{BC} (x^2 + y^2)dx - 2xydy = \int_{x=a}^{x=0} (x^2 + b^2)dx = -\frac{a^2}{3} - ab^2 \qquad \text{........(v)}$$

$$\int_{co} F.dr = \int_{co} (x^2 + y^2)dx - 2xydy = \int_{y=0}^{y=b} 0\, dy = 0 \qquad \text{.....(vi)}$$

on putting the value from (iii), (iv), (v), (vi) in (ii) we get reqd. result i.e.

$$\int_c F.dr = -2ab^2 \qquad \text{...............(A)}$$

Again curl $F = \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = -4yk$

Also **n = k**

$\therefore$ curl $F.n = (-4yk).k = -4y$

$\therefore \iint_S curl\ F.n\ ds = \int_0^a \int_0^b (-4y)dxdy$ [limit as per que.]

$\therefore \qquad = -4\int_0^a [y^2/2]_o^b dx$

$\therefore \qquad = -2b^2[x]_0^a = -2ab^2$ ....(B)

$\therefore$ From (A) and (B) we have

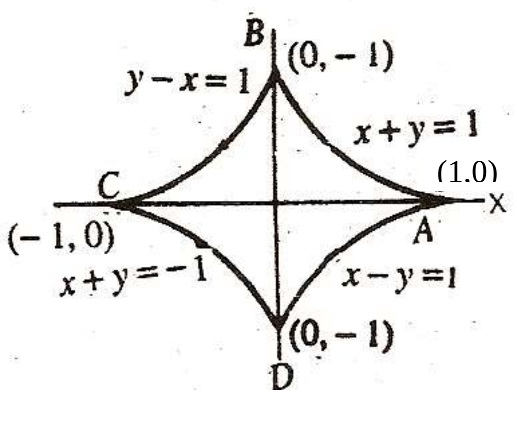
$\int_c F.dr = \int_s (curl\ F).n\,ds,$ hence theorem verified.

Ex2 Evaluate $\int_c (xydx + xy^2dy)$ where C is the square in xy - plane with vertices (1,0), (-1,0), (0,1) and (0,–1).

Sol. The given integral is $\int_c (xydx + xy^2dy) = \int_c (xyi + xy^2 j).dr$

$[since\ r = xi + yj\ \&\ dr = idx + jdy]$

$\int_c (xyi + xy^2 j).dr = \iint_s curl(xyi + xy^2 j).nds$ [By Stokes theorem]



$$\text{Now } curl F = \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)k$$

$\therefore \quad curl\ F.n = [(y^2 - x)k].k = (y^2 - x) \quad [\ \therefore\ \hat{n} = k\ ]$

$\iint_S curl\ F.n\ ds = 4 {}_0\int^1 {}_0\int^x y^2 dxdy$

[since square is symmetric in all four quadrants]

$= 4\ {}_0\int^1 xy^2 dy$

$= 4\ {}_0\int^1 y^2(1 - y)dy$

[as line passes through two points (0,1), (0-1)therefore equation of line is $x + y = 1 \Rightarrow x = 1 - y$]

$= 4\ {}_0\int^1 [y^2 - y^3]dy$

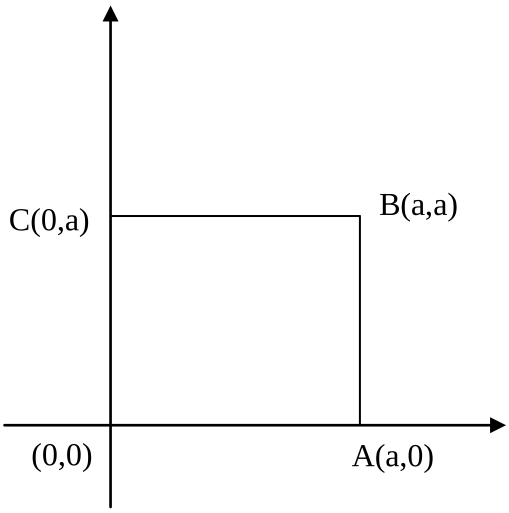
$= 4[y^3/3 - y^4/4]_0^1 = 4[1/3 - 1/4] = 1/3.$ Ans.

Ex 3 Verify Stoke's theorem for $F = x^2 i + xyj$, where C is square in *xy*-plane bounded by $x = 0, x = a, y = 0$ and y = a.

Soln. In the *xy*-plane we have

$r = xi + yj \qquad \therefore dr = idx + jdy$

$\therefore F.dr = x^2 dx + xydy$ ......(i)



clearly the path of integration C consists four straight line OA, AB, BC & CO as shown in fig.

$\therefore$ Reqd. line integral

$$\int_c F.dr = \int_{OA} F.dr + \int_{AB} F.dr + \int_{BC} F.dr + \int_{CO} F.dr \qquad \text{.........(ii)}$$

clearly on (1) OA $y = 0 \Rightarrow dy = 0$ and x varies from 0 to a

(2) AB $x = a \Rightarrow dx = 0$ and y varies from 0 to a

(3) BC $y = a \Rightarrow dy = 0$ and x varies from a to 0

(4) CO $x = 0 \Rightarrow dx = 0$ and y varies from a to 0

$$\therefore \int_{OA} F.dr = \int_{OA} x^2 dx + xydy = \int_{x=0}^{x=a} x^2 \, dx = \frac{a^3}{3} \qquad \text{.........(iii)}$$

$$\int_{AB} F.dr = \int_{AB} x^2 dx + xydy = \int_{y=0}^{y=a} ay \, dy = \frac{a^3}{2} \qquad \text{.........(iv)}$$

$$\int_{BC} F.dr = \int_{BC} x^2 dx + xydy = \int_{x=a}^{x=0} x^2 dx = -\frac{a^3}{3} \qquad \text{.........(v)}$$

and $\int_{cO} F.dr = \int_{CO} x^2 dx + xydy = \int_{y=a}^{y=0} 0 dy = 0$ ....(vi)

on putting the value from (iii), (iv), (v), (vi) in (ii) we get the reqd. result i.e.

$$\int_c F.dr = \frac{a^3}{2} \qquad \text{.....(A)}$$

Again $curl\, F = \begin{vmatrix} i & j & k \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x^2 & xy & 0 \end{vmatrix} = i(0) + j(0) + k(y) = yk$

Also n = k

$\therefore curl\ F.n = (yk).k = y$

$\therefore \iint_S curl \quad F.n \ ds = \int_0^a \int_0^a y \ dxdy$ [limit as per que.]

$$\therefore \qquad = \int_0^a \left[\frac{y^2}{2} \right]_o^a \text{dx}$$

$$\therefore \qquad = \left(\frac{a^2}{2} \right) [x]_0^a = \frac{a^3}{2} \qquad \text{....................(B)}$$

$\therefore$ From (A) and (B) we have

$$\int_c F.dr = \int_s (curl\ F).\ n\ ds, \text{ hence theorem verified.}$$

<u>2.5 Green's Theorem</u>: If c be the regular closed curve in *xy*-plane bounding a region R and P(*x*,*y*) and Q(*x*′, *y*′) be continuous on C and inside it, having continuous partial differentiation then

$$\int_c (Pdx + Qdy) = \iint_s (\delta Q/\delta x - \delta p/\delta y)dxdy \qquad ....(i)$$

Working rule: Step(1) Compare given integral in the question with the L.H.S. of (i), find P, Q and integration limit from the region R.

Step(2) Find $\delta Q/\delta x$, and $\delta P/\delta y$ by differentiating partially.

Step(3) Put all the values in right hand side of (i).

Step(4) On solving we get required result.

<u>Solved Examples</u>

Ex. 1 by using Green's theorem evaluate $\int_c [(\cos x \ \sin y - xy)dx + \sin x \cos y dy]$, where C is the circle $x^2 + y^2 = 1$.

Soln. Here P = $(\cos x \sin y - xy)$, *Q=sin x cos y* and the integration limit is $x = -1$ to $x = 1$ and $y = -\sqrt{(1-x^2)}$ *to* $y = \sqrt{(1-x^2)}$

$\therefore$ $\delta P/\delta y = \delta(\cos x \sin y - xy)/\delta y = \cos x \cos y - x$

[on differentiating w.r.t. y treating x as constant]

similarly $\delta Q/\delta x = \delta \sin x \cos y / \delta x = \cos x \cos y$

[on differentiating w.r.t. *x* treating *y* as constant]

Now by Green's theorem we have

$$\int_c (pdx + Qdy) = \iint_s (\delta Q/\delta x - \delta P/\delta y)dxdy$$

$$= \iint_s (\cos x \cos y - \cos x \ \cos y + x)dxdy$$

$$=_{x=-1}\int^{x=1}{}_{y=-\sqrt{(1-x^2)}}\int^{y=\sqrt{(1-x^2)}} x\,dx\,dy$$

$$=2\ {}_{x=-1}\int^{x=1} x[y]_0^{\sqrt{(1-x^2)}}\,dx$$

$$=2\ {}_{x=-1}\int^{x=1} x\sqrt{(1-x^2)}\,dx=0$$ [by definite integral property] Ans.

Ex.2 by using Green's theorem evaluate $\int_c \left[\left(e^{-x}\sin y\right)dx+e^{-x}\cos y dy\right]$, where C is a rectangle whose vertices are $(0,0),(\pi,0),(\pi,\pi/2) and (0,\pi/2)$.

Sol. Here P = e^{-x} sin y, Q = e^{-x} *cosy* and the integration limit is $x=0$ *to* $x=\pi$ and $y=0$ to $y=\pi/2$.

$\therefore\ \ \delta P/\delta y=\delta\left(e^{-x}\sin y\right)/\delta y=e^{-x}\cos y$

[on differentiating w.r.t. y treating x as constant]

Similarly $\delta Q/\delta x=\delta e^{-x}\cos y/\delta x=-e^{-x}\cos y$

Now by Green's theorem we have

$$\int_c (pdx+Qdy)=\iint_s (\delta Q/\delta x-\delta P/\delta y)dxdy$$

$$=\iint_s \left(-e^{-x}\cos y-e^{-x}\cos y\right)dxdy$$

$$=-2\int_{x=0}^{x=\pi}\int_{y=0}^{y=\pi/2} e^{-x}\cos y\ dxdy$$

$$=-2\left[-e^{-x}\right]_0^{\pi}\left[\sin y\right]_0^{\pi/2}$$

$$=2\left(e^{-\pi}-1\right).$$ Ans.

Ex 3 Evaluate by Green's theorem $\int_c \left[\left(xy+y^2\right)dx+x^2dy\right]$, where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

Sol. here $P=xy+y^2, Q=x^2$

Therefore $\delta P/\delta y=\delta\left(xy+y^2\right)/\delta y=x+2y$

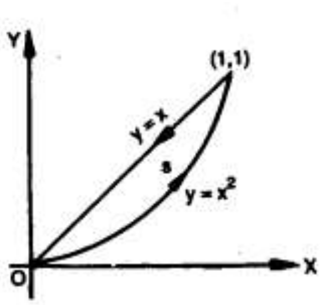
And $\delta Q / \delta x = \delta x^2 / \delta x = 2x$

Also from the given region, integration limits are $x = 0$ to 1 and $y = x^2$ to x

Now by Green's theorem we have

$$= \int_c (Pdx + Qdy) = \iint_S (\delta Q / \delta x - \delta P / \delta y) dxdy$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (2x - [x + 2y]) dxdy$$



$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (x - 2y) dxdy$$

$$= \int_{x=0}^{x=1} \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{x=1} \left[x^4 - x^3\right] dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

= -1/20 Ans.

CHECK YOUR PROGRESS:

By using Stoke's theorerm

Q.1 Verify Stokes theorem when F = yi + zj + xk and surface S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.

Q.2 Verify Stokes theorem for F = $(x^2 + y^2)i - 2xyj$ and s is the surface of the rectangle bounded by x = -a, x = a, y = 0, y = b.

By using Green's theorem

Q.3 Find the value of the integral $\int_c [(-y \sin x) dx + \cos x dy)]$, where C is the triangle formed by lines $y = 0, x = \pi / 2, and y = 2x / \pi$.

Q.4 Find the value of the integral $\int_c [(x^2 + y)dx + (2x + y)dy]$, where C is the square formed by lines x = 0, x = 1, y = 0, and y = 1.

****

UNIT : 3

GEOMETRY

UNIT- 3

BLOCK INTRODUCTION

Geometry is a very important part of Mathematics , but at the same time it is very difficult for a common student to understand the basics.

In this block we have tried to explain the concept in easy manner to understand the reader followed by solved examples.

Unit - III We find certain crucial points which lies on conic and are responsible to determine the shape. We illustrated tracing of conic with the help of few examples. In the second phase ideas of confocal conics is given.

Object : At the end of this block a reader would be able to apply the ideas in solving the related problems.

UNIT 3

UNIT III : SYSTEM OF CONICS

STRUCTURE

3.1 General equation of second degree

3.2 Tracing of conics

3.3 System of conics : Confocal conics

3.4 Polar Equation of a conic

3.1 General Equation of Second Degree

The general equation of second degree is given by

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \qquad \ldots\ldots\ldots\ldots(1)$$

We know that equation (1) always represent a conic section i.e. (1) may either be a parabola or an ellipse or a hyperbola or a circle or a pair of straight lines.

3.1.1 Nature of a conic

In general we can find the nature of a conic by the following conditions.

S. No.	Condition	Curve
1	$\Delta \neq 0, h^2 < ab$	an ellipse
2	$\Delta \neq 0, h^2 = ab$	a Parabola
3	$\Delta \neq 0, h^2 > ab$	a hyperbola
4	$\Delta \neq 0, h^2 > ab, a + b = 0$	a rectangular Hyperbola
5	$\Delta = 0$	Pair of straight lines
6	$\Delta \neq 0$ h =0, a = b	A circle
7	$\Delta = 0$, h = ab	Two parallel straight lines.

Where : $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

3.2 Tracing of conic -

The following points are useful to trace a conic.(ellipse & hyperbola)

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(1) **Nature of conic :** as per the article 3.11.

(2) **Centre :** By solving $\delta s/\delta x = 0, \delta s/\delta y = 0$

(3) **Determination of c' :** Find $c' = gx' + hy' + c$, where x', y' be the centre of conic.

(4) **Standard form :** Transform the equation of conic in standard form $Ax^2 + 2Hxy + By^2 + c' = 0.$

(5) **Length of the axes :** $(A - 1/r^2)(B - 1/r^2) = H^2$, find two values of r i.e. r_1& r_2.

(6) **Equation of axes :** Equation of axes are $(A - 1/r_1^2)x + Hy = 0$ & $(A - 1/r_2^2)x + Hy = 0$ and change in reference of origin.

(7) **Eccentricity :** $e = \sqrt{(1 - r_2^2/r_1^2)}$

(8) **Latus rectum :** $1 = 2r_2^2/r_1$

(9) **Foci** : Co-ordinates of foci $(x' + er_1\cos\alpha,\ y' + er_1\sin\alpha).$,

Here $\tan\alpha = \dfrac{\left(A - \dfrac{1}{r_1^2}\right)}{H}$ is slope of axis.

(10) **Directix :** Equation of directrix is obtained by $(x - x')\cos\alpha + (y - y')\sin\alpha = \pm r_1/e$

(11) Equation of asymptotes : (In case of a Hyperbola)

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy - g\alpha - f\beta = 0$$

Where $\alpha = \dfrac{hf - bg}{ab - h^2}$, $\beta = \dfrac{gh - af}{ab - h^2}$

3.2.1 Tracing of parabola

Let the parabola be given

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ......(i)$$

Since (i) represents a parabola, so the terms of second degree in (i) must form a perfect square, therefore put $a = a^2, b = \beta^2$ so that $h^2 = ab = a^2\beta^2$ i.e. and reduce (i) to $(ax + \beta y)^2 + 2gx + 2fy + c = 0$ which can be written as

$(ax + \beta + \lambda)^2 = 2x(\alpha\lambda - g) + 2y(\beta\lambda - f) + (\lambda^2 - c)$. (Note) .... (ii)

Choose λ so that straight line $\alpha x + \beta y + \lambda = 0$

and $2x(\alpha\lambda - g) + 2y(\beta\lambda - f) + (\lambda^2 - c) = 0$ are at right angles to each other.

There fore, $\left(-\dfrac{\alpha}{\beta}\right)\left(-\dfrac{\alpha\lambda - g}{\beta\lambda - f}\right) = -1$ or $\lambda = \dfrac{\alpha f + \beta g}{\alpha^2 + \beta^2}$ ....(iii)

Putting this value of λ (ii), we get

$(\alpha x + \beta y + \lambda)^2 = \dfrac{2(\alpha f - \beta y)}{\alpha^2 + \beta^2}(\beta x - \alpha y) + \lambda^2 - c$ (Note)

Which can be written as

$$\left[\frac{\alpha x + \beta y + \lambda}{\sqrt{(\alpha^2 + \beta^2)}}\right]^2 = \frac{2(\alpha f - \beta g)}{\sqrt{(\alpha^2 + \beta^2)}}\left[\frac{\beta x - \alpha y + k}{\sqrt{\alpha^2 + \beta^2}}\right]\frac{1}{\alpha^2 + \beta^2}$$

or $\left[\dfrac{\alpha x + \beta y + \lambda}{\sqrt{(\alpha^2 + \beta^2)}}\right]^2 = \dfrac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{3/2}}\left[\dfrac{\beta x - \alpha y + k}{\sqrt{(\alpha^2 + \beta^2)}}\right]$

or $\qquad Y^2 = \dfrac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{3/2}} X$, ...(iv)

Where $Y = (\alpha x + \beta y + \lambda)/\sqrt{\alpha^2 + \beta^2}$; $X = (\beta x - \alpha y + k)/\sqrt{(\alpha^2 + \beta^2)}$

The equation (iv) is of the form $Y^2 = 4aX$.

Hence latus rectum of the parabola = $4a = \frac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{3/2}}$

Axis of the parabola is $Y = 0$ i.e. $\alpha x + \beta y + \lambda = 0$ ....(v)

and the tangent at the vertex of the parabola is

$X = 0$ i.e. $\beta x - \alpha y + k = 0$ ...(vi)

Vertex of the parabola is obtained by solving X=0 and Y=0 i.e. equation (v) and (vi).

Directix of the parabola is given by X+a =0

Equation of the latus rectum is given by X-a=0.

Focus of the parabola is obtained by solving X=a, Y=0.

Tracing : Draw the rectangular axes and plot the vertex. Draw the axis and tangent at the vertex, which are mutually perpendicular lines through the vertex. Find also the points where the parabola meets the co-ordinate axes. Then trace the curve.

SOLVED EXAMPLE

Ex.1 : Trace the conic $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$ and find the co-ordinates of its focus and the equation of its latus rectum.

Sol. Here $a = 16, b = 9, h = -12, g = -52, f = -86, c = 44$

therefore : $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 16 & -12 & -52 \\ -12 & 9 & -86 \\ -52 & -86 & 44 \end{vmatrix} \neq 0$

Also $h^2 = ab$ so the conic is a parabola Now by rearrangement of the equation we have

$(4x - 3y)^2 - 104x - 172y + 44 = 0$ ....(1)

This can be written as

$$(4x-3y+\lambda)^2=(8\lambda+104)x+(-6\lambda+172)y+(\lambda^2-44) \quad \text{................(ii)}$$

Choose λ such that the straight lines $4x-3y+\lambda=0$ and

$(8\lambda+104)x+(-6\lambda+172)y+(\lambda^2-44)=0$ are at right angles. So $m_1.m_2 = -1$

Or $(4/3)[(-8\lambda+104)/(-6\lambda+172)]=-1$ *or* $25\lambda=50$ *or* $\lambda=2.$

Hence equation (ii) reduces to $(4x-3y+2)^2=120x+160y-40$

Or $(4x-3y+2)^2=40(3x+4y-1)$

Or $25\left[(4x-3y+2)/\sqrt{4^2+3^2)}\right]^2=40\times 5\left[(3x+4y-1)/\sqrt{(4^2+3^2)}\right]$ (Note)

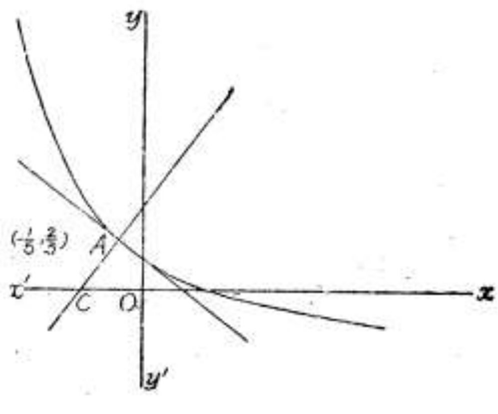
$[(4x-3y+2)/5]^2=8[(3x+4y-1)/5]$ *or* $Y^2=8X$

where $Y=(4x-3y+2)/5,\ X=(3x+4Y-1)/5$

$\therefore$ latus rectum of the parabola i.e. $4a=8\Rightarrow a=2$

Axis of the parabola is $Y=0$ i.e. $4x-3y+2=0$ ……… (iii) and

Tangent at the vertex is $X=0$ i.e. $3x+4y-1=0$ …….. (iv)



Now on solving (iii) and (iv) we get the co-ordinates of the vertex of the parabola as $(-1/5,\ 2/5)$.

Also the focus of the parabola is given by $X=a,\ Y=0$ i.e. $1/5(3x+4y-1)=2$ & $4x-3y+2=0$

Solving these we find the focus is (1,2).

Therefore the equation of the latus rectum is $X - a = 0$

i.e. $1/5(3x + 4y - 1) = 2 \Rightarrow 3x - 4y - 11 = 0$

Required tracing of curve is as per diagram.

Ex. 2 : Trace the conic $8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$ and find the equation of its axes.

Sol. The conic is $8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$ …………(i)

Here $h^2 = (-2)^2 = 4$ and $ab = 8 \times 5 = 40$ and $\Delta \neq 0$

Since $h^2 < ab$ therefore the conic is an ellipse.

The centre (x', y') of this conic is given by

$\frac{\delta s}{\delta x} = 8x' - 2y' - 8 = 0$ and $\frac{\delta s}{\delta y} = -2x' + 5y' - 7 = 0$ on solving we get $x' = 3/2, y' = 2$.

$\therefore$ the centre is (3/2,2).

Referred to parallel axes through the centre, the equation of the conic (i) is $8x^2 - 4xy + 5y^2 + c' = 0$ ………..(ii)

where $c' = gx' + fy' + c = 0 = -8(3/2) - 7(2) + 17 = -9$

$\therefore$ from (ii) the equation of the conic referred to parallel axes through the centre is $8x^2 - 4xy + 5y^2 = 9$ ………….(iii)

Or $(8/9)x^2 - 2(2/9)xy + (5/9)y^2 = 1$ ………….(iv)

The direction of the axes are given by tan $2\theta = 2h/(a-b)$

Or $\lfloor 2\tan\theta/(1 - \tan^2\theta) \rfloor = -4/(8-5) = -4/3$ or $2\tan^2\theta - 3\tan\theta - 2 = 0$

Or $\tan\theta\left[3\pm\sqrt{(9+16)}\right]/4=2,-1/2$

Let $\tan\theta_1=2$ *and* $\tan\theta_2=-1/2$. Now changing (iii) to polar co-ordinates we get

$$r^2=\frac{9\left(\cos^2\theta+\sin^2\theta\right)}{8\cos^2\theta-4\cos\theta\sin\theta+5\sin^2\theta}=\frac{9\left(1+\tan^2\theta\right)}{8-4\tan\theta+5\tan^2\theta}$$

when $\tan\theta_1=2,\ r_1^2=9(1+4)/[8-4(2)+5(4)]=45/20=9/4$

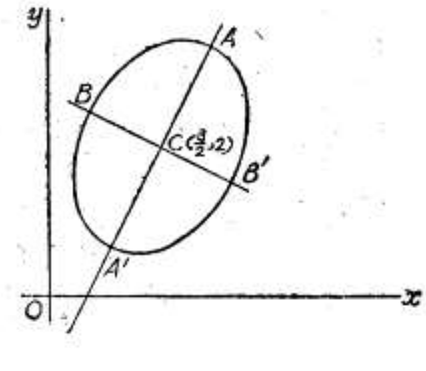
when tan $\theta_2=-1/2, r_2^2=9(1+1/4)/[8-4(-1/2)5(1/4)]=45/(32+8+5)=1$

$\therefore$ the lengths of the semi axes of the ellipse are 3/2 and 1.

Hence to draw the curve take the point $C(3/2,2)$. Through C draw ACA' inclined at an angle $\tan^{-1}2$ with x-axis and mark off $CA=CA'=3/2$.

Draw BCB' inclined at right angles to ACA' and mark off $CB=CB'=1$.
Putting $y=0$ (i) we get $8x^2-16x+17=0$, which gives imaginary values of x.
$\therefore$ the conic does not meet the x-axis.

Putting $x=0$ in (i), we get $5y^2-14y+17=0$, which also gives imaginary values of y.
$\therefore$ the conic does not meet the y-axis



The shape of the curve is as shown as in the above figure.

Equation of the axes : Consider the equation (iv) of the conic . It is in the standard form $Ax^2 + 2Hxy + By^2 = 1$

Therefore the equation of the major axis of the ellipse referred to original axes is

$(A - 1/r_2^2)(x - x') + H(y - y') = 0$ or $(8/9 - 1/1)(x - 3/2) - 2/9(y - 2) = 0$ or $2x - y - 1 = 0$

Also the equation of the minor axis of the ellipse referred to original axes is

$(A - 1/r_2^2)(x - x') + H(y - y') = 0$ or $(8/9 - 1/1)(x - 3/2) - 2/9(y - 2) = 0$ or $2x - y - 1 = 0$

Ans.

Ex.3: Trace the conic $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$. Also find its foci and eccentricity.

Sol. The conic is $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$. ....(i)

Here $h^2 = (2)^2 = 4$ and $ab = 1 \times 1 = 1$ and $\Delta \neq 0$

Since $h^2 > ab$ therefore the conic is an hyperbola.

The centre (x', y') of this conic is given by

$x' + 2y' - 1 = 0$ and $2x' + y' + 1 = 0$ on solving we get $x' = -1,\ y' = 1$.

$\therefore$ the centre is (-1,1).

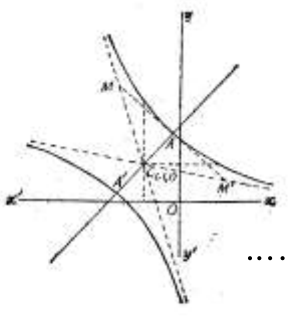
Referred to parallel axes through the centre, the equation of the conic

(i) Is $x^2 + 4xy + y^2 + c' = 0$ ....(ii)

where $c' = gx' + fy' + c = 0 - 1(-1) + 1(1) - 6 = -4$

$\therefore$ from equation (ii)

$x^2 + 4xy + y^2 = 4$ ....(iii)



The direction of the axes are given by $\tan 2\theta = 2h/(a-b)$

Or $\tan 2\theta = 4/(1-1) = \infty \; or \; 2\theta = 90^0, 270^0$

Or $\theta = 45^0, 135^0$

Let $\theta_1 = 45^0 \; and \; \theta_2 = 135^0$.

Now changing (iii) to polar co-ordinates we get

$$r^2 = \frac{4(\cos^2\theta + \sin^2\theta)}{\cos^2\theta + 4\cos\theta\sin\theta + \sin^2\theta} = \frac{4(1+\tan^2\theta)}{1+4\tan\theta+\tan^2\theta}$$

When $\theta = 45^0$, $r_1^2 = 4(1+1)/[1+4(-1)+1] = 4/3$

When $\theta = 135^0$, $r_2^2 = 4(1+1)/[1+4(-1)+1] = -4$

$\therefore$ The transverse axis is of length $4/\sqrt{3}$ making an angle of 45^o with x-axis.

Hence to draw the curve take the point $C(-1, 1)$. Through C draw a straight line ACA' inclined at an angle of 45^o with x-axis and mark off $CA = CA' = 2/\sqrt{3}$ Also draw a straight line MAM' perpendicular to CA and mark off $AM = AM' = \sqrt{4} = 2$

Join MC and M'C and produce asymptotes.

Putting x=0 in (i) ,we get $y^2 + 2y - 6 = 0$, which gives x=±√7-1=3.6 and 1.6 approx.

$\therefore$ the conic meets the x-axis at a distance 3.6 and -1.6. from the origin

Putting $y+0$ in (i), we get $x^2 - 2x - 6 = 0$, which gives $x = \pm\sqrt{7} + 1 = 3.6$ and -1.6 approx.

$\therefore$ The conic meets y-axis at distances 1.6 and-3.6 from the origin.

The shape of the curve is as shown as in the above figure.

Eccentricity: The eccentricity of the conic

$= \sqrt{(r_1^2 - r_2^2)/r_1^2} = \sqrt{[(4/3)-(-4)]/(4/3)} = \sqrt{4} = 2$ Ans.

Foci : If θ be the inclination of the transverse axis , the co-ordinates of the foci referred to the original origin are $(x' + er_1 \cos\theta, y' + er_1 \sin\theta)$

$= [-1 \pm 2(2/\sqrt{3}).(1/\sqrt{2}); 1 \pm 2(2/\sqrt{3}).(1/\sqrt{2})]$

$\theta = 45^0, r_1 = 2/\sqrt{3}$

$[-1 \pm 2(\sqrt{2}/\sqrt{3}); 1 \pm 2(\sqrt{2}/\sqrt{3})]$ Ans.

CHECK YOUR PROGRESS :

Q.1 Trace the conic $16x^2 + 24xy + 9y^2 - 3x + 4y - 7 = 0$

Q.2 Trace the conic $8x^2 + 4xy + 5y^2 - 24(x + y) = 0$

Q.3 Trace the conic $2x^2 + 5xy + 3y^2 - 9x - 11y + 10 = 0$

Q.4 Trace the conic $13x^2 - 10xy + 13y^2 + 10x - 26y - 59 = 0$

Q.5 Trace the conic $x^2 - 2xy + y^2 - 3x + y - 2 = 0$.

3.3 System of conic : Confocal conics-

Conic : The general equation of conic is given by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

where $a, b, c, f, g \,\&\, h$ are six constants. Throughout this unit we shall denote equation of conic as $S = 0$.

Equation of conic under different Conditions :

(1) Equation of conic through the point of intersection of a conic $S = 0$ and a straight line $U = 0$ is given by $S + \lambda U = 0$, where λ is an arbitrary constant.

(2) Equation of conic through the points of intersection of a conic $S=0$ and two straight lines $U_1 = 0$ and $U_2 = 0$ is given by $S + \lambda U_1 U_2 = 0$, where λ is an arbitrary constant.

(3) Equation of conic passing through the points of intersection of conics $S_1 = 0$ and $S_2 = 0$ is given by $S_1 + \lambda S_2 = 0$, where λ is an arbitrary constant.

(4) Equation of COMMON CHORD of two circles $S_1 = 0$ and $S_2 = 0$ is obtained by $S_1 - S_2 = 0$.

{Note : Before subtracting make the coefficient of x^2 and y^2 unity in both the equations.}.

(5) Equation of conic through the five points if no four points are in a straight line.

If three points are in straight line then the required conic must be a pair of straight lines.

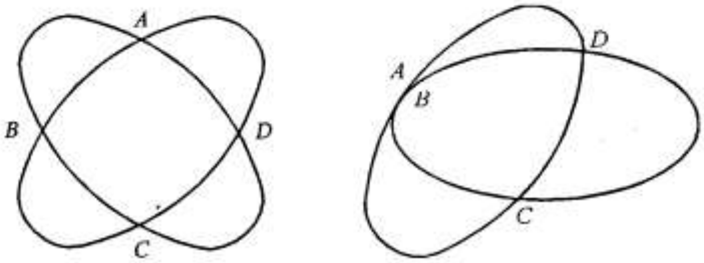
(6) Equation of conic through the four fixed points is given by

$(ax+by-1).\ (ax'+by'-1)+\lambda xy=0$

Where $ax+by-1=0$ & $ax'+by'-1=0$ are the equations of line joining two points.

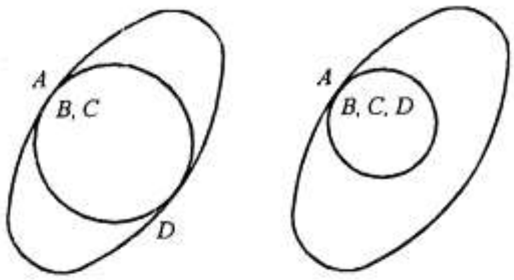
Contact of conics- let S=0 and s1=0are two conics, if we eliminate x between these two equations , the resulting equation is clearly of fourth degree and hence we conclude that two conics cuts each other at four points in which two or all four points may be imaginary.

Case I : If two conics cut each other like diag. (i), then it is known as the contact of zeroth order



Case 2 : If two conics touch each other at A intersects each other at two different points, then it called contact of first order

Case 3 : If the three intersection point of two conics coincide and fourth be the different, then it is called contact of second order



Case 4 : If the all four intersection points coincides then it is called contact of third order.

Double Contact : If two conics touch each other at two points , then they are said to have double contact. Required equation of conic having double contact with a given conic is $S+\lambda U^2=0$, where $S=0, U=0$ are the equation of conic & a straight line.

Confocal conics : Two conics are said to be confocal , when they have common foci .

Note : Equation of confocal conic with $x^2/a^2+y^2/b^2=1$ **is given by** $x^2/(a^2+\lambda^2)+y^2/(b^2+\lambda^2)=1$

Property of confocal conic : Confocal conics cut each other at right angle .

Proof : Let the two Confocal conics are $x^2/(a^2+\lambda_1)+y^2/(b^2+\lambda_1)=1$

and $x^2/(a^2+\lambda_2)+y^2/(b^2+\lambda_2)=1$

Let they cut each other at point (x_1, y_1). Now the equation of tangents to them at (x_1,y_1) are

$xx_1/(a^2+\lambda_1)+yy_1/(b^2+\lambda_1)=1$

and $xx_2/(a^2+\lambda_2)+yy_2/(b^2\lambda_2)=1$

Clearly their slopes are

$-x_1(b^2+\lambda_1)/y_1(a^2+\lambda_1)$ & $-x_1(b^2+\lambda_2)/y_1(a^2+\lambda_2)$

If they cut each other at right angles ,then

$$\frac{x_1^2(b^2+\lambda_1)(b^2+\lambda_2)}{y_1^2(a^2+\lambda_1)(a^2+\lambda_2)}=-1 \Rightarrow \frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \qquad \text{........(i)}$$

Also (x_1, y_1) lies on both the conics, therefore

$x_1^2/(a^2+\lambda_1)+y_1^2/(b^2+\lambda_1)=1$ and $x_1^2/(a^2+\lambda_2)+y_1^2/(b^2+\lambda_2)=1$

On subtracting, we get

$$x_1^2\left[1/(a^2+\lambda_1)-1/(a^2+\lambda_2)\right]+y_1^2\left[1/(b^2+\lambda_1)-1/(b^2+\lambda_2)\right]=0$$

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0 \qquad \text{.................... (ii)}$$

Since (i) and (ii) are the identical , confocal cuts each other at right angle .

SOLVED EXAMPLE

Ex.1 : Find the equation of the parabola which touches the conic

$x^2+xy+y^2-2x-2y+1=0$

at the point where it is cut by the line $x+y+1=0$.

Sol. The equation of the conic which touches the given conic at the points where it is cut by the given line is

$\left(x^2+xy+y^2-2x-2y+1\right)+\lambda(x+y+1)^2=0$ ....(i)

$(1+\lambda)x^2+(1+2\lambda)xy+(1+\lambda)y^2(2\lambda-2)x+(2\lambda-2)y+(\lambda+1)=0$ ....(ii)

If (ii) represents a parabola then we must have $"h^2=ab"$.

Or $[1/2(1+2\lambda)]^2=(1+\lambda).(1+\lambda)\Rightarrow(1+2\lambda)^2=4(1+\lambda)^2\Rightarrow\lambda=-3/4.$

$\therefore$ From (i) the required equation is

$\left(x^2+xy+y^2-2x-2y+1\right)-3/4(x+y+1)^2=0$

Or $x^2-2xy+y^2-14x-14+1=0$. .... Ans.

Ex.2: Find the equation of the circle which passes through the given point (2,-3) and the point of intersections of circles $x^2+y^2+2x+3y=7$ & $x^2+y^2-6x+2y=5$

Sol. Required equation of conic which passes through the point of intersections of two circles is given by

$\left(x^2+y^2+2x+3y-7\right)+\lambda\left(x^2+y^2-6x+2y-5\right)=0$ ....(i)

[property (i)]

As per the question conic (i) passes through the point (2,-3), therefore

On putting $x=2$ and $y=-3$ in (i), we get $\lambda=1/10$ ....(ii)

Now On putting the value of λ in (i) we get the required equation i.e.

$10\left(x^2+y^2+2x+3y-7\right)+\left(x^2+y^2-6x+2y-5\right)=0$

$\Rightarrow 11x^2+11y^2+14x+32y=75$ ....Ans.

Ex.3: Find the conic confocal with the conic $x^2+2y^2=2$ which passes through the point (1,1).

Sol.: The equation of the given conic can be written as

$x^2/2+y^2/1=1$ …………..(i)

Then the equation of its confocal conic can be written as

$$\frac{x^2}{2+\lambda}+\frac{y^2}{(1+\lambda)}=1 \qquad \text{…………(ii)}$$

since (ii) passes through the given point (1,1), therefore

$$\Rightarrow \quad \frac{1^2}{(2+\lambda)}+\frac{1^2}{(1+\lambda)}=1$$

$$\Rightarrow \quad \lambda^2+\lambda-1=0 \quad \Rightarrow \lambda_1=\left(-1+\sqrt{5}\right)/2 \;\;\&\;\; \lambda_2=\left(-1-\sqrt{5}\right)/2$$

Now $\quad 2+\lambda_1=2+\left(-1+\sqrt{5}\right)/2=\left(\sqrt{5}+3\right)/2$

and $1+\lambda_2=1+\left(-1+\sqrt{5}\right)/2=\left(\sqrt{5}+1\right)/2$

On putting the above values in (ii) we get an equation of confocal conic i.e.

$$\frac{2x^2}{\left(\sqrt{5}+3\right)}+\frac{2y^2}{\left(\sqrt{5}+1\right)}=1$$

On rationalizing the denominator & solving we get ,

$\Rightarrow 3x^2-y^2+\sqrt{5}\left(y^2-x^2\right)=2.$ ....Ans. (i)

Similarly another equation of confocal conic is obtain by putting the value of λ_2 in (ii).

Ex.4: Prove that the equation to the hyperbola drawn through point on the ellipse $x^2/a^2+y^2/b^2=1$ whose eccentric angle is 'α' and which is confocal with the ellipse is

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2$$

Sol. The equation of the given ellipse $x^2/a^2+y^2/b^2=1$ …..(i)

Equation of confocal to conic (i) is $x^2/\left(a^2+\lambda\right)+y^2/\left(b^2+\lambda\right)=1.$ …….(ii)

Now we know that the co-ordinates of the point on the ellipse $x^2/a^2+y^2/a^2+y^2/b^2=1$ whose eccentric angle is 'α' is $(a\cos\alpha,\ b\sin\alpha)$.

Since the confocal (ii) passes through point $(a\cos\alpha,\ b\sin\alpha)$, we have

$$\frac{a^2\cos^2\alpha}{(a^2+\lambda)}-\frac{b^2\sin^2\alpha}{(b^2+\lambda)}=1$$

On solving, we get

$$\lambda^2+\lambda\left(a^2\sin^2\alpha+b^2\cos^2 a\right)=0$$

Or $\lambda=0$ *or* $\lambda=-\left(a^2\sin^2\alpha+b^2\cos^2\alpha\right)$

Now on putting the value of $\lambda=-\left(a^2\sin^2\alpha+b^2\cos^2\alpha\right)$ in (ii) we get the required equation of confocal conic i.e. $\dfrac{x^2}{\cos^2\alpha}-\dfrac{y^2}{\sin^2\alpha}=a^2-b^2$.

CHECK YOUR PROGRESS :

Q.1 Find the common chord of the circles $2x^2+2y^2+14x-18y+15=0$ and $4x^2+4y^2-3x-y+5=0$ [Hint : S_1-S_2=0]

Ans. $31x-35y+15=0$

Q.2 Find the equation of the conic which passes through the five points (0,0), (2,3), (0,3), (2,5), and (4,5).

Ans. $5x^2-10xy+4y^2+20x-12y=0$

Q.3 Show that the confocal conics cut at right angles.

3.4 Polar Equation of a conic:

Definition of conic : A conic is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line . Here fixed point is known as focus , fixed straight line is directrix and constant ratio is known as eccentricity. Generally we take

S- focus(Pole),SZ- axis(initial line), LSL'- latus rectum=2l (length)

SP = r , angle PSN =θ , KZ= directrix (fixed straight line).

TO REMEMBER

Note(1) Polar equation of conic with eccentricity e and latus-rectum 2*l* is given by $l/r = 1 + e\cos\theta$ (axis is taken as initial line &pole as focus)

NOTE : Throughout this topic we shall consider the equation of conic as $l/r = 1 + e\cos\theta$.

Note : **(2)** Polar equation of conic is $l/r = 1 + e\cos(\theta - \alpha)$, If the axis inclined at an angle α to the initial line.

Note : (3) Equation of directrix is given by $l/r = 1 + e\cos(\theta - \alpha)$. In case 2 the equation of directrix is given by $l/r = e\cos(\theta - \alpha)$

MEANINGFUL WORDS:

1. **Eccentricity (e)–** Ratio of perpendicular distance of a point (say) P (lies on conic) from directrix KZ and focus S, i.e. SP/PM = e .
2. **Chord** – A straight line which cuts the conic at two points.
3. **Focal Chord -** A straight line which cuts the conic at two points & passes through the focus.
4. **Vectorial angle** – An angle which is obtain by joining the point with the pole (focus).
5. **Perpendicular focal chord** – Two focal chord perpendicular to each-other.

6. **Vectorial angles of vertices of a focal chord** – If α be the vectorial angle of a of vertices of a focal chord ,then the vectorial angle of another vertices is (π+α) [NOTE].
7. **Tangent of a conic** -A straight line which touch the conic at a point.
8. **Normal of a conic** - A straight line which is perpendicular to tangent at a point.
9. **Asymptotes** - A straight line which touch the conic at a point whose distance froe focus (pole) is infinity.
10. **Chord of contact** – If two tangents are drawn from a given point to a given conic , then the chord joining the points of contact is called chord of contact of the given point w.r.t. the given conic. Its equation is given by .

$$(l/r - e\cos\theta)\left(l/r^2 - e\cos\theta\right) = \cos(\theta - \theta')$$

Solved Example

Ex.1 Show that the equation $l/r = 1 + e\cos\theta$ and $l/r = -1 + e\cos\theta$ represent the same conic.

Soln. The given equation are $l/r = 1 + e\cos\theta$ ………..(i)

and $l/r = -1 + e\cos\theta$ …………..(ii)

Take a point P(r', θ') on (i), then r' given by $l/r' = 1 + e\cos\theta'$ ……….(iii)

Now we know that point (r', θ') can be written as $(-r', \pi + \theta')$, therefore it also lies on (i).so we have

$$-l/r' = 1 + e\cos(\theta' + \pi)$$

$$\Rightarrow l/r' = -1 + e\cos\theta' \quad \text{……………(iv)}$$

$$[\cos(\theta' + \pi) = -\cos\theta]$$

Here equation (iv) clearly the locus of (ii), Hence (i) and (ii) represent the same conic.

Ex.2 Show that in a conic the sum of the reciprocals of the segments of any focal chord of conic is constant.

Soln. Let the equation of conic be $l/r = 1 + e\cos\theta$ ....(i)

Let PSP' be a focal chord . If the vectorial angle of P is α ,

Then the vectorial angle of P' is $(\pi + \alpha)$. Now since $P(SP, \alpha)$ & $P'(SP', \pi + \alpha)$ both lies on the conic, their co-ordinates satisfies the equation (i).

Therefore we have

$l/SP = 1 + e\cos\alpha$ ....(iii)

and $l/SP' = 1 + e\cos(\pi + a) = 1 - e\cos\alpha$ ....(iv)

on adding (iii) and (iv) we get

$(l/SP) + (l/SP) = 2$

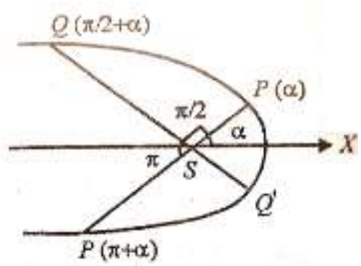
or $(1/SP) + (1/SP) = 2/l$ (constant) Proved.

Ex.3 If PSP' and QSQ' are two perpendicular focal chord of a conic , then show

That $\frac{1}{SP.SP} + \frac{1}{QS.SQ}$ Constant.

Soln. Let the equation of conic be

$l/r = 1 + e\cos\theta$ ....(i)



Let PSP' and QSQ' be two perpendicular focal chord. let the vectorial angle of P is "α"

Then the vectorial angle of P', Q and Q' Are respectively $(\pi+\alpha), \left(\frac{\pi}{2}+\alpha\right)$ and $\left(\frac{3\pi}{2}+\alpha\right)$.

Now since $P(SP,\alpha)$, $P'(SP',\pi+\alpha), Q\left(SQ,\frac{\pi}{2}+\alpha\right)$ and $Q'\left(SQ',\frac{3\pi}{2}+\alpha\right)$ lies on the conic , their co-ordinates must satisfies equation (i).

Therefore we have

$l/SP = 1+e\cos\alpha$ ....(ii)

$l/SP' = 1+e\cos(\pi+\alpha) = 1-e\cos\alpha$ ....(iii)

$l/SQ = 1+e\cos(\pi/2+\alpha) = 1+e\sin\alpha$ ....(iv)

and $l/SQ' = 1+e\cos(3\pi/2+\alpha) = 1+e\sin\alpha$ ....(v)

now multiplying (ii),and (iii) ,we get

$(l/SP)(l/SP') = (1+e\cos\alpha).(1+e\cos\alpha)$

or

$(1/SP).(1/SP') = (1-e^2\cos^2\alpha)/l^2$ .... (A)

similarly multiplying (ii),and (iii) ,we get

$(1/SQ)(1/SQ') = (1-e^2\sin^2\alpha)/l^2$ .... (B)

finally on adding (A) and (B)

that $\frac{1}{SP.SP'}+\frac{1}{QS.SQ} = \frac{(2-e)^2}{l^2}$ (constant)

Ex.4 Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

Soln. Let PSP' and QSQ' are two perpendicular focal chord of a conic, then as preceding Ex.(3), we have

$PSP' = PS+SP' = 2l(1-e^2\cos^2\alpha) = 2l/(1+2\cos^2\alpha)$ [In hyperbola $e=\sqrt{2}$]

$= -2l/\cos^2\alpha = 2l/\cos^2\alpha$ [in magnitude]

And

$$QSQ' = QS + SQ' = 2l/\left(1 - e^2 \sin^2 \alpha\right) = 2l/\left(1 - 2\sin^2 \alpha\right) = 2l/\cos^2 \alpha$$

Therefore

$PSP' = QSQ'$ Hence proved.

Ex.5 If PSQ and PHR be two chords of an ellipse through the foci S and H. Show that $(PS/SQ) + (PH/HR)$ is independent of the position of P.

Soln. Let the vectorial angle of P be ".α" and the equation of the ellipse be

$l/r = 1 + e\cos\theta$ ....(i)

since P lies on (i) ,so we get

$l/SP = 1 + e\cos\alpha$ ....(ii)

Also the vectorial angle of Q be "$\pi + \alpha$" and as Q lies on (i) , so we have

$l/SQ = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha$ ....(iii)

Adding (ii)and (iii) ,we get

$l/SP + l/SQ = 2$

or

$1/SP + 1/SQ = 2/l$

or

$1 + (SP/SQ) = 2SP/l$ (on multiplying both side by SP) ...(iv)

Similarly we can get

$1 + (PH/HR) = 2PH/l$ ....(v)

Adding (iv)and (v) ,we get

2+[(SP/SQ) + (PH/HR)] = (2/*l*) [SP+PH] =(2/*l*) (2a),

[∴ SP+PH=2a for ellipse]

or

(SP/SQ) + (PH/HR) = (4a/*l*) - 2 = constant and independent of α i.e. the position of P. **hence proved.**

Some Important Derivations :

(1) Equation of chord joining two points on a conic : let the equation of conic be $\frac{l}{r} = 1 + e\cos\theta$ ....(i)

Let α and β be the vectorial angles of the two points P and Q on (i).

Since P lies on (i) so we have

$$\frac{l}{SP} = 1 + e\cos\alpha$$

or

$$SP = \frac{l}{(1 + e\cos\alpha)}$$

∴ the polar co-ordinates of P are $\frac{l}{(1 + e\cos\alpha)}, \alpha$ Similarly

Q is $\left[\frac{l}{(1 + e\cos\beta)}, \beta\right]$

Let the polar equation of the(chord say) line PQ is

$$A\cos\theta + B\sin\theta = \frac{l}{r} \qquad \text{...........(ii).}$$

OBJECT: To find the eqn. of chord we simply find the value of A & B.

Suppose P lies on (ii) ,then we have

$$A\cos\alpha + B\sin\alpha = \frac{l}{(1 + e\cos\alpha)}$$

Or

$$A\cos\alpha + B\sin\alpha = (1 + e\cos\alpha) \text{ ....(iii)}$$

Similarly as Q lies on (ii) ,we get

$$A\cos\beta + B\sin\beta = (1 + e\cos\beta) \text{ ....(iv)}$$

On multiplying (iii) by sinβ and (iv) by cosα and subtracting we get the value of B i.e.

$$B\sin(\alpha - \beta) = \cos\beta - \cos\alpha = 2\sin\left\{\frac{1}{2}(\alpha + \beta)\right\}\sin\left\{\frac{1}{2}(\alpha - \beta)\right\}$$

Or $$B=\frac{\sin\left\{\frac{1}{2}(\alpha+\beta)\right\}}{\cos\left\{\frac{1}{2}(\alpha-\beta)\right\}}$$

Similarly on multiplying (iii) by $\sin\beta$ and (iv) by $\sin\alpha$ and subtracting we get

$$A=\frac{\cos\left\{\frac{1}{2}(\alpha+\beta)\right\}}{\cos\left\{\frac{1}{2}(\beta-\alpha)\right\}}$$

Now on substituting the values of A and B in (ii) we get

$$\frac{l}{r}=\left[\frac{\cos\left\{\frac{1}{2}(\alpha+\beta)\right\}}{\cos\left\{\frac{1}{2}(\beta-\alpha)\right\}}+e\right]\cos\theta+\left[\frac{\sin\left\{\frac{1}{2}(\alpha+\beta)\right\}}{\cos\left\{\frac{1}{2}(\alpha-\beta)\right\}}\right]\sin\theta$$

$$=$$

$$e\cos\theta+\sec\left\{\frac{1}{2}(\beta-\alpha)\right\}\left[\cos\left\{\frac{1}{2}(\alpha+\beta)\right\}\cos\theta+\sin\left\{\frac{1}{2}(\alpha+\beta)\right\}\sin\theta\right]$$

or $$\frac{l}{r}=e\cos\theta+\sec\left\{\frac{1}{2}(\beta-\alpha)\right\}\cos\left[\theta-\frac{1}{2}(\alpha+\beta)\right]$$

which is our required equation of chord.

(1) Equation of Tangent at a point $P(\alpha)$: As we know that a tangent is a straight line which touch the conic at a point whereas a chord cuts the conic at two points, therefore A CHORD BECOMES TANGENT IF 2^{ND} POINT MOVES (REACHES) TO 1^{ST} POINT. So the required equation of tangent is to be obtain by moving $Q(\beta)$ towards $P(\alpha)$. Therefore in the equation of chord above we replace β with α

$$\therefore \quad \frac{l}{r}=e\cos\theta+\sec\left\{\frac{1}{2}(\alpha-\alpha)\right\}\cos\left[\theta-\frac{1}{2}(\alpha+\alpha)\right]$$

Or $\frac{l}{r}=e\cos\theta+\cos[\theta-\alpha]$ Which is required equation of tangent.

Perpendicular Line : let the equation of a straight line is

$$A\cos\theta + B\sin\theta = \frac{l}{r} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Then equation of any line perpendicular to (i) is obtained by writing $\left(\frac{\pi}{2}+\theta\right)$ for θ and changing *l* to a new constant L. [NOTE].

(2) Equation of Normal at a pont P(α) : let the equation of conic be

$$\frac{l}{r} = 1 + e\cos\theta \qquad \ldots\ldots\ldots\text{(i)}$$

Let Equation of Tangent at a point $P(r_1,\alpha)$ to (i) is

$$\frac{l}{r} = e\cos\theta + \cos[\theta - \alpha] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Now as we know that a normal is a straight line which is perpendicular to tangent therefore with the help of the above note equation of normal is

$$\frac{L}{r} = e\cos\left(\frac{\pi}{2}+\theta\right) + \cos\left[\left(\frac{\pi}{2}+\theta\right) - \alpha\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

If (iii) is normal to (i) at $P(r_1,\alpha)$,then (iii) must pass through P. Therefore point P satisfies equation (iii) .So we have

$$\frac{L}{r_1} = e\cos\left(\frac{\pi}{2}+\alpha\right) + \cos\left[\left(\frac{\pi}{2}+\alpha\right) - \alpha\right]$$

Or $\qquad L = -er_1\sin\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$

Also $P(r_1,\alpha)$ lies on (i),therefore we have

$$\frac{l}{r_1} = 1 + e\cos\alpha$$

$\Rightarrow \qquad r_1 = \dfrac{l}{(1+e\cos\alpha)}$

on putting the value or r_1 in (iv) , we get

$$L = \frac{-el\sin\alpha}{1+e\cos\alpha} \qquad \ldots\ldots\ldots\ldots\ldots\text{(v)}$$

From (iii) and (v) equation of normal at P is

$$\left[\frac{-el\sin\alpha}{(1+e\cos\alpha).r}\right] = -e\sin\theta - \sin(\theta-\alpha)$$

(NOTE)

$$\frac{el\sin\alpha}{r.(1+e\cos\alpha)} = e\sin\theta + \sin(\theta-\alpha),$$ Which is required equation of Normal

.

Solved Example

Ex.1 If PSP' is a focal chord of a conic . Prove that the angle between tangents at P & P' is $\tan^{-1}\left\{\frac{2e\sin\alpha}{1-e^2}\right\}$ where α is a vectorial angle of P.

Soln. If vectorial angle of P is α ,then the vectorial angle of P' is (π+α).

Let the equation of conic be $\frac{l}{r} = 1 + e\cos\theta$ ………..(i)

Now equation of tangent at P(α) is

$$\frac{l}{r} = e\cos\theta + \cos[\theta-\alpha]$$

$$= e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$= (e+\cos\alpha)\cos\theta + \sin\theta\sin\alpha$$

$$l = (e+\cos\alpha)r\cos\theta + r\sin\theta\sin\alpha$$

$$l = (e+\cos\alpha)x + \sin\alpha\, y$$ [since x= rcos θ , y= rsinθ]

∴ the slope of the tangent at P is

$$m_1 = \frac{(e+\cos\alpha)}{\sin\alpha} \quad \text{(say)}$$

Now to obtain the slope of tangent at P' ,we replace α with π+α,

Therefore slope of tangent at P' is

$$m_2 = \frac{e+\cos(\pi+a)}{\sin(\pi+\alpha)} = \frac{(e-\cos\alpha)}{\sin\alpha} \text{ (say)}$$

$\therefore$ the required angle is $= \tan^{-1}\left[\frac{(m_2 - m_1)}{(1+m_1 m_2)}\right]$

$$= \tan^{-1}\left[\frac{2e\sin\alpha}{(1-e^2)}\right]$$

(on putting the values of m_1 and m_2 & solving.) hence proved.

Ex.2 Show that the condition that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may touch the conic $\frac{l}{r} = 1 + e\cos\theta$ is $(A-e)^2 + B^2 = 1$.

Soln. The equations of conic and the line are $\frac{l}{r} = 1 + e\cos\theta$ ……..(i)

$$\frac{l}{r} = A\cos\theta + B\sin\theta \quad \text{…………..(ii)}$$

The equation of the tangent to (i) at P(α) is

$$\frac{l}{r} = e\cos\theta + \cos[\theta - \alpha]$$

$$= e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$= (e+\cos\alpha)\cos\theta + \sin\theta\sin\alpha \quad \text{………..(iii)}$$

Let the line (ii) touch the conic (i) at the point P(α), then (iii) is identical to (ii).

Therefore on comparing the coefficient of the same terms ,we have

$$1 = \frac{(e+\cos\alpha)}{A} = \frac{\sin\alpha}{B}$$

$\Rightarrow$ $\cos\alpha = A - e$ and $\sin\alpha = B$

$\therefore$ on squaring and adding , we get $(A-e)^2 + B^2 = 1$. (reqd. condition)

Ex.3 Show that the two conics $\frac{l_1}{r} = 1 + e_1 \cos\theta$

and $\frac{l_2}{r} = 1 + e_2 \cos(\theta - \alpha)$ will touch one another if

$$l_1^2(1-e_2^2) + l_2^2(1-e_2^2) = 2l_1 l_2(1 - e_1 e_2 \cos\alpha)$$

Soln. Let the given conics touch one another at the point whose vectorial angle is β . Then the equations of the tangents at the common point β to the conics are

$$\frac{l_1}{r} = \cos(\theta - \beta) + e_1 \cos\theta \qquad \text{.......(i)}$$

and $$\frac{l_2}{r} = \cos(\theta - \beta) + e_2 \cos(\theta - \alpha) \qquad \text{.......(ii)}$$

equations (i) and (ii) may be written as [using cos(A-B)=cosAcosB-sinAsinB]

$$\frac{l_1}{r} = (e_1 + \cos\beta)\cos\theta + \sin\theta \sin\beta \qquad \text{........(iii)}$$

$$\frac{l_2}{r} = (e_2 \cos\alpha + \cos\beta)\cos\theta + (\sin\theta + e_2 \sin\alpha)\sin\beta \qquad \text{.......(iv)}$$

As per the question (iii) and (iv) should be identical (being common tangent)

Hence on comparing the coefficients ,we have

$$\frac{l_1}{l_2} = \frac{(\cos\beta + e_2)}{(e_2 \cos\alpha + \cos\beta)} = \frac{\sin\beta}{(\sin\theta + e_2 \sin\alpha)}$$

or

$(l_1 - l_2)\cos\beta = l_1 e_2 \cos\alpha + l_2 e_1$ and $(l_1 - l_2)\sin\beta = -l_1 e_2 \sin\alpha$

on squaring and above ,we get

$$(l_1 - l_2)^2 = l_1^2 e_2^2 + l_2^2 e_1^2 - 2 l_1 l_2 e_1 e_2 \cos\alpha$$

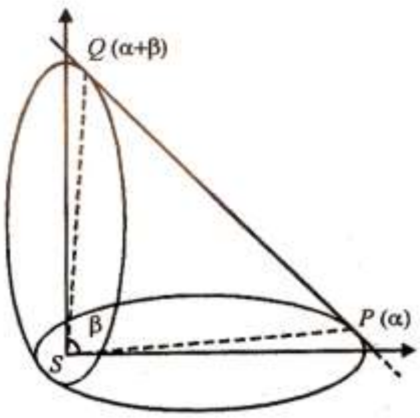
or

$$l_1^2(1-e_2^2) + l_2^2(1-e_2^2) = 2l_1 l_2(1 - e_1 e_2 \cos\alpha)$$ **Hence proved .**

Ex.4 Two equal ellipses of eccentricity e are placed with their axes at right angles and they have one focus S in common. If PQ be a common tangent, show that the angle PSQ is equal to $2\sin^{-1}\left(\frac{e}{\sqrt{2}}\right)$.

Soln. Let one of the ellipse be $\frac{l}{r}=1+e\cos\theta$ ....(i)

Then equation of the other ellipse is $\frac{l}{r}=1+e\cos(\theta-\frac{\pi}{2})$ .....(ii) [as per the given condition]



Let the common tangent PQ touch the ellipse (i) at P(α) and (ii) at Q(α'),then angle PSQ=α-α' ....(iii)

Also the tangents at P to(i) and at Q to (ii) are

$$\frac{l}{r} = e\cos\theta+\cos[\theta-\alpha]$$

$$= e\cos\theta+\cos\theta\cos\alpha+\sin\theta\sin\alpha$$

$$= (e+\cos\alpha)\cos\theta+\sin\theta\sin\alpha \quad , \qquad \text{...........(iv)}$$

and $$\frac{l}{r}= e\cos\left(\theta-\frac{\pi}{2}\right)+\cos[\theta-\alpha']$$

$$= e\sin\theta+\cos\theta\cos\alpha'+\sin\theta\sin\alpha',$$

$$= \cos\theta\cos\alpha'+(e+\sin\alpha')\sin\theta, \qquad \text{........(v)}$$

Now as (iv) and (v) represent the common tangent, so it is identical. Therefore on comparing , we get

$$1=\frac{(e+\cos\alpha)}{\cos\alpha'}=\frac{\sin\alpha}{(e+\sin\alpha')}$$

$\Rightarrow \quad (e+\cos\alpha)=\cos\alpha'$ and $\quad \sin\alpha=(e+\sin\alpha')$ ……..(vi)

Now on eliminating e , we have $\cos\alpha - \sin\alpha' = \cos\alpha' - \sin\alpha$

Or $\qquad \cos\alpha-\cos\alpha'=\sin\alpha'-\sin\alpha$

Or $\qquad 2\sin^{1/2}(\alpha+\alpha')+\sin^{1/2}(\alpha'-\alpha)=2\cos^{1/2}(\alpha+\alpha')\sin^{1/2}(\alpha'-\alpha)$

Or $\qquad \tan^{1/2}(\alpha+\alpha')=1$ or $\tan^{1/2}(\alpha+\alpha')=\tan\pi/4$

$$\Rightarrow \quad \frac{1}{2}(\alpha+\alpha')=\pi/4 \qquad \text{……….(vii)}$$

Also from (vi) we get $e = \sin\alpha - \sin\alpha'$

$= 2\cos\pi/4 \sin½ (\alpha'-\alpha)$

$= 2 (1/\sqrt{2}) \sin½ (\alpha'-\alpha)$

$\therefore$ anglePSQ = $(\alpha'-\alpha) = 2 \sin^{-1}(e/\sqrt{2})$ **Hence proved.**

CHECK YOUR PROGRESS :

Q.1 Show that the equations $l/r = 1- e\cos\theta$ and $l/r = -1- e\cos\theta$ represent the same conic.

Q.2 In any conic prove that the sum of the reciprocals of the segments of any focal chord is constant.

Q.3 PSP’ is a focal chord of the conic. Prove that the tangents at P and P’ intersects on the directrix.

Q.4 Prove that the portion of the tangent intercepted between the conic and the directrix subtends a right angle at the corresponding focus.

Q.5 Let LSL’ be the latus rectum of the conic. If the normal at L meets the conic again at Q , then show that $SQ = l(1+3e^2+e^4)/(1+e^2-e^4)$.

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UNIT - IV

CONE AND CYLINDER

Structure:

4.1 Cone with given base.

4.2 Generators of the cone.

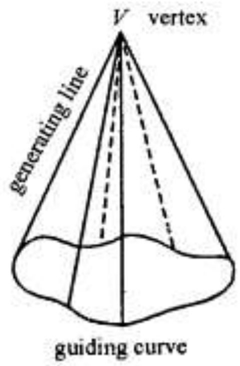
4.3 Condition for three mutually perpendicular generators.

4.4 Right circular cone.

4.5 Cylinder

4.1 Cone with given base.

4.1.1. Definition of Cone: A cone is a surface generated by moving straight line which passes through a fixed. point and intersects a given base curve. Here moving straight line is known as generating line, fixed point is called vertex.



4.1.2 Methods to find the equation of cone under different conditions:

Type I - When vertex (α, β, γ), base curve of the conic $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and equation of plane z = 0 (say) are given.

Working rule:

Step I: Consider a line passes through the vertex (α, β, γ) i.e.

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad .....(ii)$$

assuming that the l,m,n are the d.c.'s.

Step II: Find out the point of intersection of line (ii) and plane by putting z = 0 (as per the question) in (ii). So we get $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

Step III: Satisfy the equation of base curve by putting $x = \alpha - \frac{l\gamma}{n}, y = \beta - \frac{m\gamma}{n}, z = 0$

i.e. $f\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}\right) = 0 \qquad .....(iii)$

Step IV: Obtain required equation of cone by eliminating l,m,n using equation (ii)

i.e. put $\frac{l}{n} = \frac{x-\alpha}{z-\gamma}, \frac{m}{n} = \frac{y-\beta}{z-\gamma}$ [from (ii)] in (iii)

Ex. 1 Find the equation of a cone whose vertex is the point (α, β, γ) and whose generating lines passes through the conic

$$x^2/a^2 + y^2/b^2 = 1,\ z = 0.$$

Sol. The equation of the given base curve is

$$x^2/a^2 + y^2/b^2 = 1,\ z = 0.$$

The equation of any line through the vertex (α, β, γ) are

$$(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n.$$

The line (2) meets the plane $z=0$ at the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n} \text{ i.e. at the point } \left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right).$$

If this point lies on the conic (1), then

$$\frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)=1.$$

Eliminating *l, m, n* between the equations (2) and (3), the required equation of the conic is given by

$$\frac{1}{a^2}\left\{\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\}^2+\frac{1}{b^2}\left\{\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\}^2=1$$

or $$b^2(\alpha z-\gamma x)^2+a^2(\beta z-\gamma y)^2=a^2b^2(z-\gamma)^2.$$ **Ans.**

Ex.2 Find the equation of the cone with vertex (5, 4, 3) and with

$3x^2+2y^2=6, y+z=0$ **as base.**

Sol. The equations of the base curve are (given)

$$3x^2+2y^2=6, \text{ and } y+z=0.$$

The equations of any line (generator) through (5, 4, 3) are

$$\frac{x-5}{1}=\frac{y-4}{m}=\frac{z-3}{n}=\frac{y+z-7}{m+n}$$

The line (2) meets the plane $y+z=0$ at the point given by

$$\frac{x-5}{1}=\frac{y-4}{m}=\frac{z-3}{n}=\frac{0-7}{m+n}=\frac{-7}{m+n}$$

i.e. at the point $\left(5-\frac{7l}{m+n},4-\frac{7m}{m+n},3-\frac{7n}{m+n}\right)$

or $$\left(\frac{5m+5n-7l}{m+n},\frac{4n-3m}{m+n},\frac{3m-4n}{m+n}\right)$$

If this point lies on the base curve given by (1), then this point will satisfy $3x^2 + 2y^2 = 6$ and therefore

$$3(5m+5n-7l)^2+2(4n-3m)^2=6(m+n)^2.$$

Putting proportionate values of l,m,n from (2) in (3) and thus eliminating l,m,n between the equation (2) and (3), the required equation of the cone is given by

$$3\{5(y-4)+5(z-3)-7(x-5)^2\}+2\{4(z-3)-3(y-4)\}^2=6\{(y-4)+(z-3)\}^2$$

or $$3(5y+5z-7x)^2+2(4z-3y)^2=6(y+z-7)^2$$

or $$147x^2+87y^2+101z^2+90yz-210zx-210xy+84y+84z-294=0.$$ **Ans.**

Type II - When a given general equation of 2nd degree $f(x,y,z)=0$ representing a cone .

Working rule:

Step I: Make the given equation $f(x,y,z)=0$ homogeneous by introducing a new variable t. i.e. F(x,y,z,t) = 0 .....(ii)

Step II: Find $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0 \; and \; \frac{\partial F}{\partial t}=0$

Step III: Put t = 1 in all the four relation.

Step IV: Solve first three relations for *x* ,*y* and *z*.

Step V: If the obtained values of *x,y,z* satisfy fourth relation i.e. $\frac{\partial F}{\partial t}=0$ then the given equation represents a cone.

To find vertex.

Step VI: The obtained values of *x,y,z* denote the coordinates of vertex.

Ex.3 Prove that the equation

$$4x^2-y^2+2xy-3yz+12x-11y+6z+4=0.$$

represents a cone. Find the co-ordinates of its vertex.

Sol. Let

$$F(x,y,z)\equiv 4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0 \quad \text{....(1)}$$

Making (1) homogeneous with the help of t, we get

$$F(x,y,z,t)\equiv 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2 \quad \text{....(2)}$$

Differentiating (2) partially with respect to *x, y, z* and *t* successively, we have

$$\left.\begin{array}{l}\partial F/\partial x=8x+2y+12t, \partial F/\partial y=-2y+2x-3z-11t \\ \partial F/\partial z=4z+3y+6t, \partial F/\partial t=12x-11y+6z+8t\end{array}\right\} \quad \text{......(A)}$$

Putting $t=1$ in each of the relations in (A) and then equation them to zero, we get

$8x+2y+12=0$ .....(3), $2x-2y-3z-11=0$ .....(4)

$-3y+4z+6=0$ .....(5), $12-11y+6z+8=0$ .....(6)

Now we shall find *x, y, z* by solving the equations (3), (4) and (5).

Eliminating *x* between (3) and (4), we get

$$5y + 6z + 28 = 0.$$

Solving (5) and (7), we get

$$y = -2,\ z = -3.$$

Ex.4 Prove that the equation

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a cone if $u^2/a + v^2/b + w^2/c = d$.

Sol. Let

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0. \qquad ...(1)$$

Making (1) homogeneous with the help of *t*, we get

$$F(x, y, z, t) \equiv ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2wzt + dt^2. \qquad ...(2)$$

Differentiating (2) partially w.r.t. *x, y, z* and *t* successively, we have

$$\left.\begin{array}{l}\partial F/\partial x = 2ax + 2ut, \partial F/\partial y = 2by + 2vt \\ \partial F/\partial z = 2cz + 2wt, \partial F/\partial t = 2ux + 2vt + 2wz + 2dt\end{array}\right\} \qquad ...(A)$$

Putting $t = 1$ in each of the relations in (A) and then equating them to zero, we have

$$2ax + 2u = 0, 2by + 2v = 0,\ 2cz + 2w = 0,$$

$$2ux + 2vy + 2wz + 2d = 0.$$

From the first three equations, we have

$$x = -u/a,\ y = -v/b,\ z = -w/c.$$

Putting these values of *x, y, z* in the fourth equation namely 2ux + 2vy + 2wz + 2d = 0, we have

2u (-u /a) + 2v (-v/b) + 2w (-w/c) + 2d = 0

or $$u^2/a + v^2/b + w^2/c = d\ . \qquad ...(3)$$

Hence (3) is the required condition that the given equation (1) represents a cone.

Important Note:

Every second degree homogeneous equation whose vertex is at origin represent a cone. Therefore the equation $ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0$ represent a cone with vertex (0,0,0).

Type III - Equation of a cone whose vertex is at the origin and equation of base curve and the plane is given or two equations representing the guiding (base) curve in which one equation is of first degree.

Working rule:

Required equation of cone is obtained by making the other equation homogeneous with the help of the first equation.

Ex. 5 Find the equation of the cone with vertex at (0,0,0) and passing through the circle given by

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0, x - y + z = 2.$$

Sol. The given equations of circle are

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0 \quad .....(1)$$

and $$x - y + z = 2 \; or \; (x - y + z)/2 = 1 \quad .....(2)$$

Making (1) homogeneous with the help of (2), the equation of the required cone with the vertex at the origin is

$$(x^2 + y^2 + z^2) + \frac{1}{2}(x - 2y + 3z)(x - y + z) - 4.\frac{1}{4}(x - y + z)^2 = 0$$

or $$2(x^2 + y^2 + z^2) + (x^2 - 3xy + 4zx + 2y^2 - 5yz + 3z^2)$$

$$-(x^2 + y^2 + z^2 - 2xy + 2zx - 2yz) = 0$$

or $$x^2 + 2y^2 + 3z^2 + xy - yz^2 = 0$$ **Ans.**

Ex.6 Find the equation of the cone whose vertex is (0,0,0) and which passes through the curve of intersection of the plane.

$$lx + my + nz = p$$

and the surface $$ax^2 + by^2 + cz^2 = 1$$

Sol. The equation of the given curve are

$$ax^2 + by^2 + cz^2 = 1 \quad .....(1)$$

and $$(lx + my + nz)/p = 1 \quad .....(2)$$

Making (1) homogeneous with the help of (2), the equation of the required cone with the vertex at the origin is given by

$$(ax^2 + by^2 + cz^2) = \{(lx + my + nz) / p)^2$$

or $$p^2(ax^2 + by^2 + cz^2) = (lx + my + nz)^2.$$ **Ans.**

4.2 Generators of the Cone:

Definition: A straight line which generates the cone is called generating line or generators.

4.2.1 Finding generator lines.

Type IV: To find the equation of generating line when the equation of cone and equation of plane is given.

Working rule

Step I: Consider a straight line (by assuming it as reqd. equation of generating line)

i.e. $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ .....(i)

Step II: Since equation (i) is a generators of the given cone, therefore its d.c.'s l,m,n satisfy the equation of cone and plane. So put $x = l$, $y = m$ and $z = n$ in both the equations and numbering (ii) and (iii).

Step III: Eliminate m [(or n), (or l)] between (ii) and (iii).

Step IV: Make the obtained equation quadratic in terms of $\frac{l}{n}\left[\left(or\frac{m}{n}\right),\left(or\frac{l}{m}\right)\right]$

Step V: Finally by putting the values of l & n in (ii) [equation of plans]. We get the values of l,m,n.

So the required equation of generators is obtained by putting the values of l,m,n in (i).

Ex.7 Find the angle between the lines of section of the plane 2x + y + 5z = 0 and the cone 6*yz* - 2*zx* + 5*xy* = 0.

Sol. The equation of the given plane is

$3x + y + 5z = 0$ ...(1)

The equations of given cone s $6yz - 2zx + 5xy = 0$. ...(2)

Let the equation of a line of section of the cone (2) by the plane (1) be given by $x/l = y/m = z/n$ ...(3)

Then $3l + m + 5n = 0$ .....(4) and $6mn - 2nl + 5lm = 0$ ...(5)

Eliminating m between (4) and (5), we get

$6(-3l-5n)n - 2nl + 5l(-3l-5n) = 0$ or $30n^2 + 45nl + 15l^2 = 0$

or $2n^2 + 3nl + l^2 = 0$ or $(2n + l)(n + 1) = 0$.

$\therefore$ $2n + l = 0$ or $n + 1 = 0$.

When $2n + l = 0$ *i.e.* $l = -2n$ then from (4), $m = n$.

$\therefore$ $l/(-2) = m/1 = n/1$.

Again when $n + l = 0$ *i.e.* $l = -n$ then from (4), $m = -2n$.

$\therefore$ $l/1 = m/2 = n/(-1)$.

Hence the equations of the lines of section are

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{1} \text{ and } \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}.$$

Let θ be the angle between these lines of section.

Then $\cos\theta = \dfrac{(-2).1 + (1).(2) + (1).(-1)}{\sqrt{\{(-2)^2 + (1)^2 + (1)^2\}}\sqrt{\{(1)^2 + (2)^2 + (-1)^2\}}} = -\dfrac{1}{6}$.

$\therefore$ the acute angle $\theta = \cos^{-1}(1/6)$.

4.2.2 Angle between two generators line.

Let the two generators line of a cone are $\dfrac{x}{l_1} = \dfrac{y}{m_1} = \dfrac{z}{z_1}$ *and* $\dfrac{x}{l_2} = \dfrac{y}{m_2} = \dfrac{z}{n_2}$ then angle between two generators lines is given by

$$Cos\theta = \frac{l_1.l_2 + m_1m_2 + n_1n_2}{\sqrt{(l_1^2 + m_1^2 + n_1^2}.\sqrt{(l_2^2 + m_2^2 + n_2^2}}$$

NOTE: Two generating lines are said to be mutually perpendicular if angle between them is $\pi/2$ or Cos $\pi/2$ =0 Or $l_1l_2 + m_1m_2 + n_1n_2 = 0$

Ex.8 Show that the plane *ax* + *by* + *cz* = 0 cuts the cone *yz* + *zx* + *xy* = 0 in perpendicular lines if 1/*a* + 1/*b* + 1/*c* = 0.

Sol. Let the equations of a line of section of the section of the cone $yz + zx + xy = 0$ by the plane $ax + by + cz = 0$ be given by

$$x/l = y/m = z/n \quad ...(1)$$

Then $mn + nl + lm = 0$ ...(2)

and $al + bm + cn = 0$. ...(3)

Eliminating n between the equations (2) and (3), we get 0

$$m\{-(al + bm)/c\} + \{-(al + bm)/c\}\, l + lm = 0$$

or $al^2 + (a + b - c)\, lm + bm^2 = 0$, or $a(l/m)^2 + (a+b-c)(l/m) + b = 0$. ...(4)

This is a quadratic equation in l/m and hence it shows that the given plane cuts the given cone in two generators (i.e. lines). Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of these two lines so that the product of the roots of the equation (4) is given by

$$\frac{l_1}{m_1}.\frac{l_2}{m_2} \leq \frac{b}{a}.$$

$$\therefore \quad \frac{l_2 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \text{ (by symmetry).}$$

There lines of section will perpendicular if

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ *i.e.if* $1/a + 1/b + 1/c = 0$.

4.3 Condition for three mutually perpendicular generators.

A cone whose vertex is at the origin having three mutually perpendicular generators if the sum of the coefficient of the terms x^2, y^2 and z^2 is zero i.e. if the equation of cone is.

$ax^2 + by^2 + cz^2 + 2xyz + 2gzx + 2hxy = 0$ the required condition for three mutually perpendicular generators is a + b + c = 0

Ex.9 If $x/1 = y/2 = z/3$ represents one of a set of three mutually perpendicular generators of the cone 5*yz* - 8*zx* - 3*xy* = 0, find the equations of the other two.

Sol. The equation of the given cone is.

$$5yz - 8zx - 3xy = 0 \quad \text{……………..(1)}$$

In the equation of the cone (1), the sum of the coefficients of x^2, y^2 and z^2 is zero, hence the cone (1) has an infinite set of three mutually perpendicular generators.

Thus if $x/1 = y/2 = z/3$ is one of a set of three mutually perpendicular generators then the other two generators will be the lines of intersection of the given cone (1) by the plane through the vertex (3, 0, 0) and perpendicular to the given generator namely $x/1 = y/2 = z/3$ i.e. by the plane $1.x + 2.y + 3.z = 0$. Let $x/l = y/m = z/n$ be a line of intersection, so that we have

$$5mn - 8nl - 3lm = 0 \quad \text{...(2)}$$

and $$l + 2m + 2n = 0 \quad \text{...(3)}$$

Eliminating l between (2) and (3), we get

$5mn - 8n(-2m - 3n) - 3(-2m - 3n)m = 0$

or $6m^2 + 30mn + 24n^2 = 0$ or $m^2 + 5mn + 4n^2 = 0$

or $(m + 4n)(m + n) = 0$ or $m = -4n$, $m = -n$.

When $m = -4n$, from (3). we have $l - 5n = 0$ or $l = 5n$.

$\therefore$ $4l = -5m = 20n$ or $l/5 = m/(-4) = n/1$ ...(4)

When $m = -n$, from (3), we have $l + n = 0$ or $l = -n$.

$l = m = -n$ or $l/1 = m/1 = n/1\,(-1)$ ...(5)

Therefore, from (4) and (5), the equations of the other two generators are

$x/5 = y/(-4) = z/1$ and $x/1 = y/1 = z/(-1)$ ...(6)

Clearly both these generators are perpendicular since

$5.1 + (-4).1 + (1).(-1) = 0$.

Also each of these two generators is perpendicular to the given generator $x/1 = y/2 = z/3$ since

$5.1 + (-4).2 + 1.3 = 0$ and $1.1 + 1.2 + (-1).3 = 0$.

Hence the required two generators are given by the equations (6) above.

Ex.10 Show that the condition that the plane $ux + vy + wz = 0$ may cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators is

$$(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$$

Sol. This problem has already been solved in deduction 2 of 10. But here we provide its complete solution.

Let the equations of a line of section of the cone $ax^2 + by^2 + cz^2 = 0$ by the plane $ux + vy + wz = 0$ be

$x/l = y/m = z/n$. .....(1)

Then $al^2 + bm^2 + cn^2 = 0$ .....(2)

and $ul + vm + wn = 0$. ....(3)

Eliminating n between (2) and (3), we get

$$al^2 + bm^2 + c\{-(ul + vm)/w\}^2 = 0$$

or $$(aw^2 = cu^2)l^2 + 2culm + (bw^2 + cv^2)m^2 = 0$$

or $$(aw^2 + cu^2)(l/m)^2 + 2cuv(l/m) + (bw^2 + cv^2) = 0 \;...(4)$$

This is a quadratic equation in l/m and hence it shows that the plane $ux + vy + wz = 0$ cuts the given cone into two lines (or generators). Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of these lines. Then l_1/m_1 and l_2/m_2 are the roots of the equation (4). The product of the roots of the equation (4) is given by

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{bw^2+cv^2}{aw^2+cu^2}$$

$\therefore$ $$\frac{l_1 l_2}{bw^2+cv^2}=\frac{m_1 m_2}{cu^2+aw^2}=\frac{n_1 n_2}{av^2+bu^2}$$

writing the third fraction by symmetry.

Now the generators (i.e. the lines) will be perpendicular if

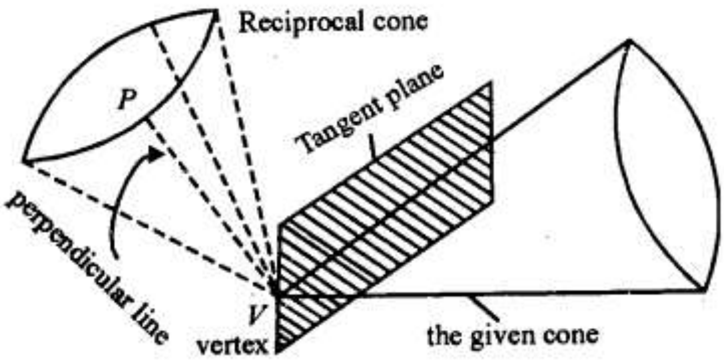
$$l_1 l_2+m_1 m_2+n_1 n_2=0$$

i.e. if $$(bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0$$

i.e. if $$(b+c)u^2+(c+a)v^2+(a+b)w^2=0.$$ **Proved**

4.3.1 The reciprocal cone

Definition: The reciprocal cone of the given cone is the locus of the lines through the vertex and at right angles to the tangent planes of the given cone.



Working rule to find the equation of reciprocal cone of a given cone -

$$ax^2+by^2+cz^2+2fyz+2gzx+2bxy=0 \qquad .....(1)$$

Step I: First assume that the equation of reciprocal cone is

$$Ax^2+By^2+Cz^2+2Fzx+2Gzx+2Hzy=0 \qquad .....(2)$$

Step II: Compare the given equation of cone with (1) and find the value of a,b,c,f,g,h.

Step III: Find the values of A,B,C,F,G,H where

$$A=\begin{vmatrix} b & f \\ f & c \end{vmatrix}, B=\begin{vmatrix} a & g \\ g & c \end{vmatrix}, C=\begin{vmatrix} a & h \\ h & b \end{vmatrix}, F=-\begin{vmatrix} a & h \\ g & f \end{vmatrix}, G=\begin{vmatrix} h & b \\ g & f \end{vmatrix}, H=-\begin{vmatrix} h & g \\ f & c \end{vmatrix}$$

Step IV: To obtain the equation, put all the values in (2).

Ex.11 Find the equation of the cone reciprocal to the cone.

$fyz + gzx + hxy = 0$

Sol. The equation of the given cone is

$$fyz + gzx + hxy = 0. \qquad .....(1)$$

Let the reciprocal cone of (1) be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0. \qquad .....(2)$$

Comparing the equation (1) with the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

we have $a = 0, b = 0, c = 0,\ f = \frac{1}{2} f,\ g = \frac{1}{2} g$ *and* $h = \frac{1}{2} h.$

$\therefore$ $$A = "bc - f^2" = \frac{1}{2} f^2; B = -\frac{1}{4} g^2; C = -\frac{1}{4} h^2;$$

$$F = "gh - af" = \frac{1}{2} g.\frac{1}{2} h - 0 = \frac{1}{4} gh; G = \frac{1}{4} hf; H = \frac{1}{4} fg.$$

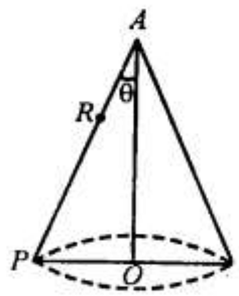
Putting these value in (2), the equation of the cone reciprocal to (1) is given by

$$-\frac{1}{4} f^2x^2 - \frac{1}{4} g^2y^2 - \frac{1}{4} h^2z^2 + 2.\frac{1}{4} ghyz + 2.\frac{1}{4} hfzx + 2.\frac{1}{4} fgxy = 0$$

or $$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0$$

4.4 Right Circular Cone:

Definition: A right circular cone is a surface generated by a line which moves in such a way that it passes through a fixed point (vertex) and it makes a constant angle θ with a fixed straight line through the vertex.



Here angle θ is known as semi-vertical angle and the fixed straight-line through the vertex is called the axis.

(B) To find the equation of a right circular cone.

Let $A(\alpha, \beta, \gamma)$ be the vertex of the cone an the equations of the axis AO be

$$(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n,$$

where *l*, *m*, *n* are the d.r.'s of the axis. Let the co-ordinates of any point R on the surface of the cone be (*x*, *y*, *z*) so that the d.r.'s of the line *AR* are $x-\alpha,\ y-\beta,\ z-\gamma.$ If θ is the semi-vertical angle of the cone *i.e.* the angle between the lines *AO* and *AR* is θ, then it is given by

$$\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{\{(x-\alpha)^2+(y+\beta)^2+(z-\gamma)^2\}}}$$

Squaring and cross-multiplying, the required equation of the right circular cone is given by

$$\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2$$
$$=(1^2+m^2+n^2)\{(x-\alpha)^2+(y-\beta)^2+(z+\gamma)^2\}\cos^2\theta.$$

Case I. If the vertex is at origin. If the vertex A be taken at (0, 0, 0) then putting $\alpha=\beta=\gamma=0$ in the equation (1) above, the equation of the right circular cone with the vertex at the origin an semi-vertical angle θ is given by

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta$$

or $$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)(1-\sin^2\theta)$$

or $$(l^2+m^2+n^2)(x^2+y^2+z^2)\sin^2\theta=(l^2+m^2+n^2)(x^2+y^2+z^2)-(lx+my+nz)^2$$

or $$(l^2+m^2+n^2)(x^2+y^2+z^2)\sin^2\theta$$
$$=(mz-ny)^2+(nx-lz)^2+(lz-mx)^2. \qquad .....(2)$$

[By Lagrange's Identity]

Case II. If the vertex is at the origin, the axis of the cone is the z-axis and the semi-vertical angle is θ.

Let *OX*, *OY*, *OZ* be the co-ordinate axes. The vertex of the cone is at the origin *O*, the axes of the cone is along z-axis *OZ* and the semi-vertical angle is θ Let *R(x,y,z)* be any current point on the surface of the cone so that *OR* is a generator of the cone.

The d.c.'s of *OZ* are 0,0,1 and the d.r.'s of OR are x – 0, y – 0, z – 0 i.e. *x,y,z*. The angle between the lines *OR* and *OZ* is equal to the semi-vertical angle θ of the cone. Therefore

$$\frac{x.0+y.0+z.1}{\sqrt{(0^2+0^2+1^2}\sqrt{x^2+y^2+z^2}}=\frac{z}{\sqrt{(x^2+y^2+z^2}}$$

or $$z^2\sec^2\theta=x^2+y^2+z^2 \quad\text{or}\quad x^2+y^2=z^2(\sec^2\theta-1)$$

or $$x^2 + y^2 = z^2 \tan^2 \theta \quad .....(3)$$

Case III: If the vertex is at the origin, axis the y-axis and semi vertical angle is θ.

The d.c.'s of the y-axis are 0,1,0.Hence putting $l = n = 0$ and $m = 1$ in the equation (2) above, the required equation of the cone is given by

$$(x^2 + y^2 + z^2)\sin^2 = z^2 + x^2, \text{ or } (z^2 + x^2)(1 - \sin^2 \theta) = y^2 \sin^2 \theta$$

or $$z^2 + x^2 = y^2 \tan^2 \theta \quad .....(4)$$

Case IV: If the vertex is at the origin, axis the x-axis and semi vertical angle is θ.

The d.c.'s of the x-axis are 1,0,0.Hence putting $m = n = 0$ and $l = 1$ in the equation (2) above, the required equation of the cone is given by

$$(x^2 + y^2 + z^2)\sin^2 \theta = z^2 + y^2, \text{ or } (y^2 + z^2)(1 - \sin^2 \theta) = x^2 \sin^2 \theta$$

or $$y^2 + z^2 = x^2 \tan^2 \theta \quad .....(5)$$

Note: The students are advised to write complete proofs of cases III and IV as given in the proof of case II.

Ex.12 Find the equation to the right circular cone whose vertex is (2,–3,5), axis makes equal angles with the co-ordinates axes and semi-vertical angle is 30^0.

Sol. The co-ordinates of the vertex A of the cone are given as (2,–3,5). If *l,m,n* are the d.c.'s of the axis of the cone then since the axis makes equal angles with the co-ordinate axes, we have

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{(l^2 + m^2 + n^2}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

Consider a general point *R(x,y,z)* on the cone so that the d.r.'s of the generator AR are $x - 2$, $y + 3$, $z - 5$.

Now the semi-vertical angle of the cone is 30^0 i.e. the angle between the axis of the cone and the generator *AR* is 30^0, therefore, the required equation of the right circular cone is given by

$$\cos 30^0 = \frac{(1/\sqrt{3}).(x-2) + (1/\sqrt{3})(y+3) + (1/\sqrt{3})(z-5)}{\sqrt{\{(x-2)^2 + (y+3)^2 + (z-5)^2\}}}$$

or $$\frac{\sqrt{3}}{2}=\frac{x+y+z-4}{\sqrt{3}\sqrt{\{(x-2)^2+(y+3)^2+(z-5)^2\}}}$$

Squaring and cross multiplying, we get.

$$9\{x^2-4x+4+y^2+6y+9+z^2-10z+25\}=4(x^2+y^2+z^2$$
$$+16+2xy+2xz+2yz-8x-8y-8z\}$$

or $$5(x^2+y^2+z^2)-8(yz+zx+xy)-4x+86y-58z+278=0$$

Ex.13 A right circular cone is passing through the point (1,1,1) and its vertex is the point (1,0,1). The axis of the cone is equally inclined to the co-ordinate axes. Find the equation of the cone.

Sol. The cone is passing through the point (1,1,1) and hence the d.r.'s of the generator passing through (1,0,1) and (1,1,1) are 1, –1, 1–1, 1–1, i.e. 0, 1, 0.

The axis of the cone is equally inclined to the co-ordinate axes and so its d.c.'s are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ [See above example]

Thus if θ be the semi-vertical angle of the cone, then

$$\cos\theta=\frac{0.(1/\sqrt{3}+1.(1/\sqrt{3})+0.(1/\sqrt{3})}{\sqrt{(0^2+1^2+0^2)}}=\frac{1}{\sqrt{3}} \qquad .....(1)$$

Now let $R(x,y,z)$ be a general point on the cone and so the d.r.'s of the generator AR are x–1, y–0, z–1. The angle between the axis of the cone and the generator AR is also θ and so, we have

$$\cos\theta=\frac{(x-1).(1/\sqrt{3})+y.(1/\sqrt{3})+(z-1).(1/\sqrt{3})+}{\sqrt{\{(x-1)^2+(y)^2+(z-1)^2\}}}$$

or $$\frac{1}{\sqrt{3}}=\frac{x+y+z-2}{\sqrt{3}\sqrt{(x^2+y^2+z^2-2x-2z+2}}, \qquad \text{using (1)}$$

Squaring and cross multiplying, we get

$$(x^2+y^2+z^2-2x-2z+2)=(x+y+z-2)^2$$

or $$yz+zx+xy-x-2y-z+1=0$$

This is required equation of the right circular cone.

Ex.14 Find the equation of the right circular cone with vertex at (1,–2,–1), semi-vertical angle 60^0 and the axis.

$(x-1)/3=-(y+2)/4=(z+1)/5.$

Sol. The vertex of the cone is $A(1,-2,1)$. The equation of the axis of the cone $(x-1)/3=-(y+2)/4=(z+1)/5.$

$\therefore$ The d.r.'s of the axis of the cone are 3,–4,5. The semi vertical angle of the cone is 60^0.

Consider a general point $R(x,y,z)$ on the cone and so the d.r.'s of the generator AR are $x-1$, $y+2$, $z+1$. Hence the required equation of the right cone is given by

$$\cos 60^0 = \frac{3.(x-1)+(-4)(y+2)+5.(z+1)}{\sqrt{\{(3)^2+(-4)^2+(5)^2\}}\sqrt{\{(x-1)^2+(y+2)^2+(z+1)^2\}}}$$

or $$\frac{1}{2}=\frac{3x-4y+5z-6}{\sqrt{(50)}\sqrt{(x^2+y^2+z^2-2x+4y+2z+6)}}$$

Squaring and cross-multiplying, we get

$$25(x^2+y^2+z^2-2x+4y+2z+6)=2(3x-4y+5z-6)^2$$

or $$7x^2-7y^2-25z^2+80yz-60zx+48xy+22x+4y+170z+78=0.$$

Ex.15 Find the equation of the cone generated by rotating the line x/1 = y/m = z/n about the line *x/a = y/b = z/c* as axis.

Sol. The equation oft he axis of the cone are

$x/a = y/b = z/c$. .....(1)

The equation of a generator of the cone are

$x/l = y/m = z/n$. .....(2)

Let θ be the semi-vertical angle of the cone then it is the angle between the lines (1) and (2) and so we have

$$\cos\theta=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2}\sqrt{(l^2+m^2+n^2)}}$$

Consider a general point $R(x,y,z)$ on the cone. Now the vertex A of the cone being the point of intersection of (1) and (2) is given by (0,0,0). Thus the d.r.'s of the generator AR are $x-0$, $y-0$, $z-0$, i.e. x,y,z. Also the angle between the axis (1) and the generator AR is θ and hence, we have

$$\cos\theta=\frac{ax+by+cz}{\sqrt{(a^2+b^2+c^2)}\sqrt{(x^2+y^2+z^2)}} \quad .....(4)$$

Equating the two values of $\cos\theta$ given by (3) and (4), the required equation of the cone is given by

$$\frac{ax+by+cz}{\sqrt{x^2+y^2+z^2}}=\frac{al+bm+cn}{\sqrt{(l^2+m^2+n^2)}}$$

or $\quad (ax+by+cz)^2(l^2+m^2+n^2)=(x^2+y^2+z^2)(al+bm+cn)^2.$

CHECK YOUR PROGRESS :

Q.1 Find the equation of cone whose vertex is (α,β,γ) and the base curve is $ax^2+by^2=1, z=0$.

[Ans. $a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$]

Q.2 Find the equation of cone whose vertex is (1,2,3) and base curve is $y^2=4ax, z=0$

[Ans. $(2z-3y)^2=4a(z-3)(z-3x)$]

Q.3 Prove that the equation

$x^2-2y^2-3z^2-4xy+4yz-5zx+8x-19y-2z=20$

represent a cone whose vertex is (1,–2,3).

Q.4 Find the equation of the cone whose vertex is the origin and base curve is given by $ax^2+by^2=2z, lx+my+nz=p$

[Ans. $p(ax^2+by^2)=2z(lx+my+nz)$]

Q.5 Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $ax^2-y^2+3z^2=0$. Also find the angle between the lines of section.

[Ans. $\frac{x}{1}=\frac{y}{-4}=\frac{z}{-2},\frac{x}{1}=\frac{y}{-2}=\frac{z}{0},\theta=Cos^{-1}\sqrt{\frac{27}{35}}$]

Q.6 Find the equation of right circular cone whose vertex is (1,1,1), axis is $\frac{x-1}{-1}=\frac{y-1}{2}=\frac{z-1}{3}$ and the semi vertical angle is 30^0.

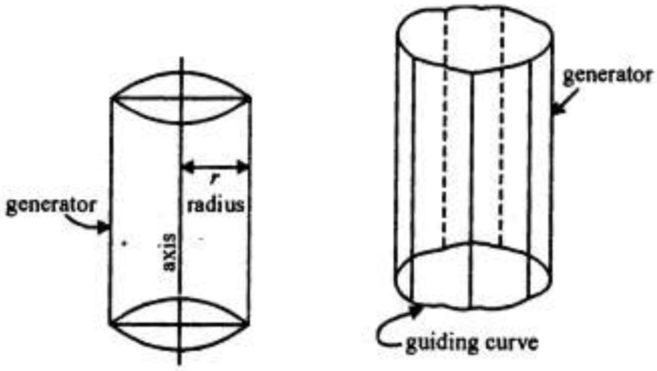
[Ans. $19x^2+13y^2+3z^2+8xy+12xz-24yz-58x-10y+6z+31=0$]

Q.7 Find the equation of the cone formed by rotating the line $2x+3y=5, z=0$ about the y-axis.

[Ans. $4x^2-9y^2+4z^2+36y-36=0$]

4.5 Cylinder

Definition: A cylinder is a surface generated by a moving straight line which moves parallel to a fixed straight line and intersects a given curve. Here the given curve is known as guiding curve, fixed straight line is axis of the cylinder.



4.5.1 To find the equation of cylinder whose generators are parallel to the line (axis) $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and guiding curve $ax^2+2hxy+zy^2+2gx+2fy+c=0$, z = 0 is given.

Working rule

Step I: Consider the equation of generating line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}, \qquad .....(1)$$

where point (x_1, y_1, z_1) lies on the cylinder.

Step II: Find the point of intersection of line (1) and plane by putting z=0 in (1) .

Step III: Point of intersections satisfy the equations of conic (guiding curve). So satisfy it.

Step IV: Finally the locus of point (x_1, y_1, z_1) gives us the required equation of cylinder. So replace x_1, y_1, z_1 by x,y,z respectively.

Ex.1 Find the equation of a cylinder whose generators are parallel to the line $x = y/2$ = –z and whose guiding curve is

$$3x^2+2y^2=1, z=1$$

Or

Find the equation of a cylinder whose generators are parallel to the line $x = y/2 = -z$ and passing through the curve.

$$3x^2 + 2y^2 = 1, z = 0$$

Sol. The equation of the given guiding curve are

$$3x^2 + 2y^2 = 1, z = 0$$

The equation of the given line is

$x/1 = y/2 = z/(-1)$ .....(2)

Consider a current point $P(x_1,y_1,z_1)$ on the cylinder. The equations of the generator through the point $P(x_1,y_1,z_1)$ which is a line parallel to the given line (2) are

$(x - x_1)/1 = (y - y_1)/2 = (z - z_1)/(-1)$ .....(3)

The generator (3) meets the plane $z = 0$ in the point given by

$$\frac{x - x_1}{1} = \frac{y - y_1}{2} = \frac{0 - z_1}{-1} \text{ i.e. } (x_1 + z_1, y_1 + 2z_1, 0)$$

Since the generator (3) meets the conic (1), hence the point $(x_1 + z_1, y_1 + 2z_1, 0)$ will satisfy the equations of the conic given by (1), and so we have

$$3(x_1 + z_1)^2 + 2(y_1 + 2z_1) = 1$$

or $3(x_1^2 + 2x_1z_1 + z_1^2) + 2(y_1^2 + 4y_1z_1 + 4z_1^2) = 1$

or $3x_1^2 + 2y_1^2 + 11z_1^2 + 8y_1z_1 + 6z_1x_1 - 1 = 0$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ i.e. the required equation of the cylinder is given by

$$3x^2 + 2y^2 + 11z^2 + 8yz + 6zx - 1 = 0$$

Ex.2 Find the equation of the cylinder which intersects the curve $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$ and whose generators are parallel to the axis of x.

Sol. The equation of the guiding curve are

$ax^2 + bu^2 + cz^2 = 1$ .....(1)

and $lx + my + nz = p$ .....(2)

Now the equation of the cylinder whose generators are parallel to x-axis will not contain the terms of x. Hence the required equation of the cylinder is obtained by eliminating x between the equation (1) and (2), and so is given by

$$a\left\{\frac{p - my - nz}{l}\right\}^2 + by^2 + cz^2 = 1$$

$$\left[\because From(2), x = \frac{p - my - nz}{l}\right]$$

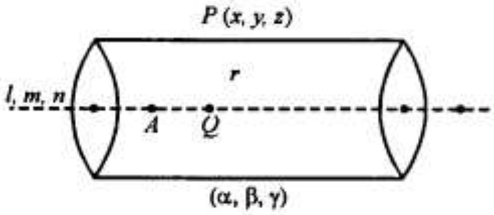
or $$a(p - my - nz)^2 + bl^2y^2 + cl^2z^2 = l^2$$

or $$a(p^2 + m^2y^2 + n^2z^2 - 2pmy - 2pnz + 2mnyz) + bl^2y^2 + cl^2z^2 = l^2$$

or $$(am^2 + bl^2)y^2 + (an^2 + cl^2)z^2 + 2amnyz - 2ampy - 2ampz + (ap^2 - l^2) = 0$$

4.5.2 Right Circular Cylinder:

Definition: A cylinder is called a right circular cylinder if the generators are always at a constant distance from the fixed straight line. Here the constant distance is called the radius.



4.5.2.1 To find the equation of right circular cylinder with radius r and equation of its axis be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

Working rule

Step I: Find AP by using the formula of a distance between two pts. A and P i.e.

$$AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-y)^2}$$

Step II: Find AQ the projection of the point A (α, β, γ) and $P(x,y,z)$ i.e.

$$AQ = \frac{(x-\alpha)l + (y-\beta)m + (z-\gamma)n}{\sqrt{(l^2 + m^2 + n^2)}}$$

Step III: Put the values in

$$AP^2 = AQ^2 + PQ^2$$

or $PQ^2 = AP^2 - AQ^2$

Where PQ is radius.

In brief compare the equation of axis with $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and find the values of $\alpha,\beta,\gamma,l,m,$ and n. After putting all the values in above, we get the required equation of a right circular cylinder.

Ex.3 Find the equation of the right circular cylinder of radius 2 whose axis passes through (1,2,3) and has direction cosines proportional to 2,–3,6.

Sol. The axis of the cylinder passes through (1,2,3) and has d.r.'s 2,–3,6 hence its equations are

$$(x-1)/2=(y-2)/(-3)=(z-3)/6 \qquad .....(1)$$

Consider a point $P(x,y,z)$ on the cylinder. The length of the perpendicular from the point $P(x,y,z)$ to the given axis (1) is equal to the radius of the cylinder i.e. 2. Hence the equation of the required cylinder is given by [Put l=2, m = –3, n = 6; α = 1, β = 2, γ = 3 and r = 2 in the equation (3) of the cylinder].

$$\{6(y-2)-(-3)(z-3)\}^2+\{2(z-3)-6(x-1)\}^2$$
$$+\{(-3)(x-1)-2(y-2)\}^2=(2)^2\{(2)^2+(-3)^2+(6)^2\}$$

or $$(6y+3z-21)^2+(2z-6x)^2+(-3x-2y+7)^2=4(49)$$

or $$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0$$

Ex.4 Find the equation of the right circular cylinder of radius 2 and having as axis the line.

$$(x-1)/2=(y-2)=(z-3)/2.$$

Sol. The equation of the axis of the cylinder are

$$(x-1)/2=(y-2)/1=(z-3)/2 \qquad .....(1)$$

Consider a point P(x,y,z) on the cylinder. The length of the perpendicular from the point P(x,y,z) to the given axis (1) is equal to the radius of the cylinder i.e. 2. Hence the required equation of the cylinder is given by

$$\{2(y-2)-1.(z-3)\}^2+\{2(z-3)-2(x-1)\}^2+\{1(x-1)-2(y-2)^2\}^2$$
$$=(2)^2.\{(2)^2+(2)^2+(2)^2\}$$

or $$(2y-z-1)^2+(2z-2x-4)^2+(x-2y+3)^2=36$$

or $5x^2+8y^2+5z^2-4yz-8zx-4xy+22x-16y-14z-10=0$

Ex.5 Find the equation of the right circular cylinder whose axis is x-2=z, y = 0 and which passes through the point (3,0,0)

Sol. The equations of the axis of the cylinder may be written as

$$(x-2)/1=(y-0)/0=(z-0)/1.$$

First we shall find the radius *r* of the cylinder.

We know that

r = the length of the perpendicular from a point (3,0,0) on the cylinder to the axis (1).

$$\frac{1}{\sqrt{\{(1)^2+(0)^2+(1)^2\}}}\sqrt{[(0-0.0)^2+\{1.0-1.(3-2)\}^2}+\{0.(3-2)-1.0\}^2]$$

$=1/\sqrt{2}$.

Consider a point *P*(*x*,*y*,*z*) on the cylinder. The length of the perpendicular from the point *P*(*x*,*y*,*z*) to the given axis (1) is equal to the radius of the cylinder. Hence the required equation of the cylinder is given by.

$$\{1.y-0.z\}^2+\{1.z-1.(x-2)\}^2+(0.(x-2)-1.y\}^2=(1/\sqrt{2})^2(1+0+1)$$

or $y^2+(z-x+2)^2+y^2=1$

or $x^2+2y^2+z^2-2zx-4x+4z+3=0$

Ex.6 Find the equation of the right circular cylinder which passes through the circle $x^2+y^2+z^2=9, x-y+z=3.$

Sol. The equation of the guiding circles are

$$x^2+y^2+z^2=9 \qquad .....(1)$$

and $x-y+z=3.$ .....(2)

Since the cylinder is a right circular cylinder, hence the axis of the cylinder will be perpendicular to the plane (2) of the circle and so the d.r.'s of the axis of the cylinder are 1,-1,1.

Let $P(x_1,y_1,z_1)$ be a point on the cylinder. The equations of the generator through $P(x_1,y_1,z_1)$ having d.r.'s 1, –1, 1 are

$$(x-x_1)/1=(y-y_1)/(-1)=(z-z_1)/1=r \text{ (say)} \qquad .....(3)$$

(1) and (2), and so we have

$$(r+x_1)^2+(-r+y_1)^2+(r+z_1)^2=9$$

and $\quad (r+x_1)-(-r+y_1)+(r+z_1)=3$

or $\quad 3r^2+2r(x_1+y_1+z_1)+(x_1^2+y_1^2+z_1^2-9)=0 \qquad .....(4)$

$$r=\frac{1}{3}(3-x_1+y_1-z_1) \qquad .....(5)$$

Eliminating r between (4) and (5), we get

$$\frac{1}{3}(3-x_1+y_1-z_1)^2+\frac{2}{3}(3-x_1+y_1-z_1)(x_1+y_1+z_1)+(x_1^2+y_1^2+z_1^2-9)=3.$$

or $\quad x_1^2+y_1^2+z_1^2+y_1z_1-z_1x_1+x_1y_1=9$

$\therefore$ The locus of $P(x_1,y_1,z_1)$ or the required equations of the cylinder is

$$x^2+y^2+z^2+yz-zx+xy=9$$

CHECK YOUR PROGRESS :

Q.1 Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and the guiding curve is $x^2=2y^2=1, z=0$

[Ans. $3(x^2+2y^2+z^2)-2zx+8yz-3=0)$]

Q.2 Find the equation of the cylinder whose generators are parallel to the line $x=\frac{y}{-2}=\frac{z}{3}$ and the guiding curve is the ellipse $x^2+2y^2=1, z=3$

[Ans. $3x^2+6y^2+3z^2+8yz-2zx-6x-24y-16z+24=0$]

Q.3 Find the equation of the right circular cylinder of radius 2 and whose axis is the line $\frac{x-1}{2}=\frac{y}{z}=\frac{z-3}{1}$.

[Ans. $10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0$]

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UNIT V

CONICOID AND THEIR PROPERTIES

Structure

5.1 Central Conicoid and paraboloids.

5.1.1 Definition of Conicoid :- The surface represented by the general equation of second degree in three variables i.e..

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

represents conicoid

5.1.2. Different forms of conicoid -

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (an ellipsoid.)

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (One sheet hyperboloid.)

(3) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (Two sheets hyperboloid)

(4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c^2}$ (Elliptic paraboloid)

(5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c^2}$ (Hyperboloid paraboloid)

5.1.3 Tangent lines and tangent plane to a conicoid

Definition: A line which meets a conicoid in two coincident points is called the tangent line to the conicoid and the locus of the tangent lines to a conicoid at a point on it is called tangent plane at that point.

Note - If $ax^2 + by^2 + cz^2 = 1$ is the equation of conicoid and let (α, β, γ) be a point on it, then the equation of tangent plane at point (α, β, γ) is given by $a\alpha x + b\beta\gamma + c\gamma z = 1$

5.1.4 Condition of tangency

Let the equation of central conicoid is = $ax^2 + 3y^2 + cz^2 = 1$ .....(1)

and let the equation of given plane is $\equiv lx + my + nz = p$ .....(2)

Now we know that the equation of tangent plane to the conicoid (1) at point (α,β,γ)

is. $a\alpha x + b\beta y + c\gamma z = 1$ .....(3)

So if (2) is tangent plane to (1), then (2) & (3) represent the same plane, so on comparing the coefficients we get

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p} \Rightarrow \alpha = \frac{l}{ap}, \beta = \frac{m}{bp}, \gamma = \frac{n}{cp}$$

Now by our supposition point (α,β,γ) line on (1), we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

on putting the value of α,β,γ we get

$$a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1$$

$$\Rightarrow \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2 \qquad .....(4)$$

Which is required condition that the plane (2) touches to (1)

Note (1) Equation of tangent plane to conicoid (1) is obtained by putting the value of p from (4) in (2)

$$lx + my + nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

Note (2) The conditions when the plane (2) touches the Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $a^2l^2 + b^2m^2 + c^2n^2 = p^2$

and equation of tangent plane is

$$lx + my + nz = \pm\sqrt{a^2l^2 + b^2m^2 + c^2n^2}$$

Note (3) Equation of tangent plane to the paraboloid $ax^2 + by^2 = 2cz$ at the point (α,β,γ) is given by $a\alpha x + b\beta y = c(z+\gamma)$ and the condition that the plane (2) touch to the paraboloid is given by

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0$$

and equation of tangent plane is given by

$$2n(lx+my+xz)+c(\frac{l^2}{a}+\frac{m^2}{b})=0$$

Solved Examples

Based on tangent plane & condition

Ex.1 Find the equation of tangent plane to the conicoid

$2x^2-6y^2+3z^2=5 \; at\; the\; po\text{int}\,(1,0,1)$

Sol. Equation of tangent plane to the given coniocid at (α,β,γ) is

$$2\alpha x-6\beta y+3\gamma z=5$$

According to question $\alpha=1,\ \beta=0,\ \gamma=1$

Required equation is

$$2x+3z=5$$

Ex.2 Prove that the plane $3x+12y-6z-17=0$ .....(1)

touches the conicoid $3x^2-6y^2-9z^2+17=0$ .....(2)

Also find the point of contact

Sol. Equation of plane to the given conicoid at $(\alpha,\beta,\gamma)$is

$$3\alpha x-6\beta y+9\gamma z=-17 \qquad .....(3)$$

If (1) Touches to (2) then (1) and (3) represent the same plane so on comparing the coefficients we get

$$\frac{3\alpha}{3}=\frac{-6\beta}{12}=\frac{9\gamma}{-6}=\frac{17}{-17}$$

$$= \alpha=\frac{\beta}{-2}=-\frac{3\gamma}{2}=-1$$

$$= \alpha=-1,\beta=2,\gamma=2/3$$

$\therefore$ Plane (1) touches (2) at point (-1, 2, 2/3).

Ex.3 Find the equation of tangent planes to the elipsoid $7x^2+5y^2+3z^2=60$ which pass through the line $7x+10y=30, 5y-3z=0.$

Sol. Any plane through the line $7x+10y-30=0, 5y-3z=0$ is given by

$$(7x+10y-30)+\lambda(5y-3z)=0$$

or $$7x+(10+5\lambda)y-3\lambda z=30. \quad .....(1)$$

If it touches the ellipsoid

$$7x^2+5y^2+3z^2=60 \text{ or } \frac{7}{60}x^2+\frac{5}{60}y^2+\frac{3}{60}z^2=1 \quad .....(2)$$

then the condition of tangency $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p$ gives

$$\frac{60(7)^2}{7}+\frac{60(10+5\lambda)^2}{5}+\frac{60(-3\lambda)^2}{3}=(30)^2$$

or $$420+12(100+100\lambda+25\lambda^2)+20(9\lambda^2)=900$$

or $$2\lambda^2+5\lambda+3=0 \text{ or } (\lambda+1)(2\lambda+3)=0$$

or $$\lambda=-1,-\frac{3}{2}$$

$\therefore$ From (1) the required tangent planes are

$$7x+5y+3z=30; 14x+10y+9z=50$$

Ex.4 Show that the plane $x+2y-2z=4$ touches the paraboloid $3x^2+4y^2=24z$. Find the point of contact.

Sol. Let the plane $$x+2y-2z=4 \quad .....(1)$$

touches the paraboloid $$3x^2+4y^2=24z \quad .....(2)$$

at the point (α,β,γ).

The equation of tangent plane to (2) at the point (α,β,γ) is given by

$$3\alpha x+4y=12(z+y)$$

or $$3\alpha x+4y-12z=12\gamma \quad .....(3)$$

Since equation (1) and (3) represent the same plane, and so comparing (1) and (3), we get

$$\frac{3\alpha}{1}=\frac{4\beta}{2}=\frac{-12}{-2}=\frac{-12\gamma}{4} \Rightarrow \alpha=2, \beta=3, \gamma=2 \quad .....(4)$$

Again since (α,β,γ) lies on (2), we have $3\alpha^2+4\beta^2=24\gamma$ .....(5)

Since values of α,β,γ given by (4) satisfy (5), so the plane (1) touches the praboloid (2) at (α,β,γ), and from (4) the co-ordinates of point of contact are (2,3,2).

Ex. 5. Find the condition that the plane $lx + my + nz = 1$ may be a tangent plane to the paraboloid $x^2 + y^2 = 2z$

Sol. Let the plane $lx + my + nz = 1$ .....(1)

touch the paraboloid $x^2 + y^2 = 2z$ .....(2)

at the point (α,β,γ)

Now the equation of the tangent plane to (2) at the point (α,β,γ) is

$\alpha x + \beta y = z + \gamma$ or $\alpha x + \beta y - z = \gamma$ .....(3)

Since equation (1) and (3) represent the same plane, and so on comparing (1) and (3), we get

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{-1}{n} = \frac{\gamma}{1} \text{ or } \alpha = -\frac{1}{n}, \beta = -\frac{m}{n}, \gamma = -\frac{1}{n} \quad .....(4)$$

Also, since (α,β,γ) lies on the paraboloid (2), we have $\alpha^2 + \beta^2 = 2\gamma$

$$\Rightarrow \quad \left(-\frac{1}{n}\right)^2 + \left(-\frac{m}{n}\right)^2 = 2\left(-\frac{1}{n}\right), \quad \text{[From equation (4)]}$$

$$\Rightarrow \quad l^2 + m^2 = -2n \Rightarrow l^2 + m^2 + 2n = 0$$

which is the required condition.

CHECK YOUR PROGRESS :

Q.1 Find the equation of the tangent plane to the conicoid $5x^2 - 4y^2 + 6z^2 = 25$ at the point (1, -1, 2).

Q.2 Find the point of intersection of the straight line $\frac{x+5}{-3} = \frac{y-4}{1} = \frac{z-11}{7}$ and the conicoid $13x^2 - 17y^2 + 7z^2 = 7$

Q.3 Find the condition under which the plane $\alpha x + \beta y + \gamma z = 1$ may touch the conicoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Q.4 Find the equation of tangent at the point (2,0,2) to the paraboloid $x^2 + y^2 = 2z$

5.1.5 Normal - A line through a point P on a conicoid perpendicular to the tangent plane at P is called the normal to the conicoid at P.

5.1.6 General equation of the normal to the conicoid $ax^2 + by^2 + cz^2 = 1$ at any point (α,β,γ) .

Given equation of conicoid $ax^2 + by^2 + cz^2 = 1$ .....(1)

Equation of tangent plane at (α,β,γ) is $a\alpha x + b\beta y + c\gamma z = 1$ .....(2)

Now as we know that the direction cosines of the normal is proportional to $a\alpha, b\beta, c\gamma$.

$\therefore$ Equation of normal to the conicoid (1) at (α,β,γ) is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z\gamma}{c\gamma} \qquad .....(3)$$

If p is the length of the perpendicular from the origin to (2) the

$$p = \frac{1}{\sqrt{(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2}} \Rightarrow p^2 = \frac{1}{(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2}$$

$$\Rightarrow (a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1 \qquad .....(4)$$

From (3) & (4) it is clear that the direction cosines of the normal is $a\alpha p, b\beta p, c\gamma p$

$\therefore$ required equation of normal is

$$\frac{x-\alpha}{a\alpha p} = \frac{y-\beta}{b\beta p} = \frac{c-\gamma}{c\gamma p}$$

Note (1) Equation of normal to the conicoid (ellipsoid)

$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = 1$ at point (α,β,γ) is given by

$$\frac{x-\alpha}{(p\alpha/a^2)} = \frac{y-\beta}{(p\beta/b^2)} = \frac{z-\gamma}{(p\gamma/c^2)}$$

Note(2) Equation of normal to the paraboloid $ax^2 + by^2 = 2cz$ at point (α,β,γ) is given by

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c}$$

5.1.7 Number of normal drawn from a point (x_1,y_1,z_1) to a conicoid.

Let the equation of conicoid be $ax^2 + by^2 + z^2 = 1$ and let the given point be (x_1,y_1,z_1). .....(1)

Now we know that the equation of normal to conicoid (1) at point (α,β,γ) is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} \qquad .....(2)$$

Since the normal (2) passes through the point (x_1,y_1,z_1) we have

$$\frac{x_1-\alpha}{a\alpha} = \frac{y_1-\beta}{b\beta} = \frac{z_1-\gamma}{c\gamma} = r \text{ (say)} \qquad .....(3)$$

$$\alpha = \frac{x_1}{1+ar}, \beta = \frac{y_1}{1+br}, \gamma = \frac{z_1}{1+cr} \qquad .....(4)$$

From (1) and (4)

$$a\left(\frac{x_1}{1+ar}\right)^2 + b\left(\frac{y_1}{1+br}\right)^2 + c\left(\frac{z_1}{1+cr}\right)^2 = 1 \qquad .....(5)$$

Clearly equation (5) gives six values of r, hence we can determine 6 points on conicoid corresponding to each value of r.

$\therefore$ Six normal can be drawn from a point.

Note: In the same way we can prove that five normals can be drawn from a point to paraboloid $ax^2 + by^2 = 2cz$

SOLVED EXAMPLES

Ex.1 Find the equation of tangent and normal at the point (2,0,2) on the paraboloid $x^2 + y^2 = 2z$

Soln. Equation of normal at point (α,β,γ) to $x^2 + y^2 = 2z$ is

$$\frac{x-\alpha}{\alpha} = \frac{y-\beta}{\beta} = \frac{z-\gamma}{-1}$$

According to question α = 2, β = 0, γ = 2

Required equation is $\frac{x-2}{2} = \frac{y-0}{0} = \frac{z-2}{-1}$

Similarly equation of tangent is

$\alpha x + \beta y = (z + \gamma)$

$\Rightarrow 2x = z + 2$

$\Rightarrow 2x - z = 2$

Ex.2 Prove that the lines drawn from the origin parallel to the normal to the conicoid $ax^2 + by^2 + cz^2 = 1$ at points lying on its curve of intersection with the plane $lx + my + nz = p$ generate the cone

$$p^2\left(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right) = \left(\frac{lx}{a} + \frac{my}{b} + \right)$$

Sol. Let (α,β,γ) be any point on the curve

$$ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p \qquad .....(1)$$

$\therefore$ Equations of normal at the point (α,β,γ) to the conicoid $ax^2 + by^2 + cz^2 = 1$

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} \qquad ...(2)$$

$\therefore$ Equations of the line passing through origin and parallel to the normal (2) are

$$\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma} \qquad .....(3)$$

Since (α,β,γ) satisfies both the equations of (1), we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1, l\alpha + m\beta + n\gamma = p \qquad .....(4)$$

From (4), on making homogeneous to the first equation with the help of second equation, we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\frac{l\alpha + m\beta + n\gamma}{p}\right)^2$$

$\therefore$ locus of the line (3) is $p^2\left(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right) = \left(\frac{lx + my + nz}{p}\right)^2$ which is the required equation of required cone.

Ex.3 Find the equation of the normal at (4,3,5) on the paraboloid $\frac{1}{2}x^2 - \frac{1}{3}y^2 = z$.

Sol. The equation of the given paraboloid is

$\frac{1}{2}x^2 - \frac{1}{3}y^2 = z$ or $3x^2 - 2y^2 = 6z$. .....(1)

The equation of the tangent plane to the paraboloid (1) at (4,3,5) is

$3x(4) - 2y(3) = 3(z + 5)$ or $4x - 2y - z = 5.$

The required normal in the line through (4,3,5) at right angle to the tangent plane given by (2). Hence the required equation of the normal is

$$\frac{x-4}{4} = \frac{y-3}{-2} = \frac{z-5}{-1}$$

5.1.8 Plane of contact

The equation of tangent plane at the point (x', y', z') to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $axx' + byy' + czz' = 1$. So if the plane passes through a point (α, β, γ) then $a\alpha x' + b\beta y' + c\gamma z' = 1$

$\therefore$ Required plane of contact for point (α, β, γ) is given by

$a\alpha x + \beta y + c\gamma z = 1$

5.2 Plane section of conicoid

Introduction: We know that all plane section of a conicoid are conics and that sections by parallel planes are similar and similarly situated conics. If the plane passes through the centre of conicoid, the section of conicoid is said to be a central plane section.

5.2.1 Nature of plane section: Let the equation of central conicoid be $ax^2 + by^2 + cz^2 = 1$ and the plane be $lx + my + nz = p$

Then the nature of the plane section of a central conicoid is an ellipse, parabola or hyperbola according.

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} >, = or < 0$$

5.2.2 Area of the central plane sections

$$Ao = \pi r_1 r_2 = \pi \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{bcl^2 + cam^2 + abn^2}} \qquad .....(1)$$

If p is the length of the perpendicular drawn from the origin to the parallel tangent plane to conicoid $ax^2 + by^2 + cz^2 = 1$ then we have

$$p = \frac{\sqrt{l^2 / a + m^2 / b + n^2 / c}}{\sqrt{l^2 + m^2 + n^2}} \qquad .....(2)$$

$\therefore$ From (1) & (2) we have

$$Ao = \frac{\pi}{p\sqrt{abc}}$$

Note(1) - If the conicoid be an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then required area is obtain by replacing a,b,c with $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ respectively

Required area is $A_o = \frac{\pi abc}{p}$

Note(2) Area of non central plane section is to be obtained by replacing a,b,c by $\frac{a}{d^2}, \frac{b}{d^2}, \frac{c}{d^2}$ respectively in A_o where $d^2 = 1 - \frac{p^2}{p_0^2}$

Here $p_0^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$

CHECK YOUR PROGRESS :

Q.1 Find the equation of the normal at the point (4,3,3) on the paraboloid

$$\frac{x^2}{4} - \frac{y^2}{9} = z$$

Q.2 Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2az$, but if the point lies on the surface.

$27a\,(x^2 + y^2) + 8(a - z)^2 = 0$ two of them coincide.

Q.3 Find the equation of the normal at point (4,3,5) on the paraboloid

$$\frac{1}{2}x^2 - \frac{1}{3}y^2 = z$$

Q.4 Find the equation of normal to the ellipsoid $3x^2 + y^2 + 2z^2 = 7$ at the point (1,–2,0).

Q.5 Find the equation of the normal to the hyperboloid of one sheet $4x^2 - 3y^2 + z^2 = 2$ at the point (1,1,1).

5.3 Generating lines

5.3.1 Definition of Generating lines

A surface which is generated by a moving straight line is called ruled surface and the straight line is called generating line.

5.3.2 Nature of ruled surface

Surface on which consecutive generating lines intersect is called **developable surface**. Eg. cone while surface on which consecutive generating lines do not intersect is called a **skew surface** eg. hyperbolic paraboliod.

Working rule to find generators of a conicoid which passes through a point (α,β,γ).

Step I- Take an equation of straight line which passes through the point (α,β,γ) by supposing that the d.c's are l,m,n i.e.

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad .....(1)$$

Step II- Find the coordinates of a point lies on (1) by equating (1) with r (a constant) i.e.

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad ....(2)$$

$\Rightarrow (\alpha+lr,\beta+mr,\gamma+lr)$ lies on (1)

Step III- coordinates of the point satisfies the equation of conicoid (as the point lies on conicoid.) , so put

$x=\alpha+lr, y=\beta+mr, z=\gamma+nr$ in the equation of conicoid.

Step IV- Equate the coefficient of r^2 *and* r with zero.

Step V- Solve the obtained relation for *l,m,n*.

Step VI- Put the values of *l,m,n* in (2) to obtained the required generating lines.

Ex.1 Find the equation of the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$

Sol. The given hyperboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad .....(1)$$

Let l,m,n be the direction cosines of the generator. Then the equation of the generator through $(a\cos\alpha, b\sin\alpha, 0)$ are given by

$$\frac{x-a\cos\alpha}{l}=\frac{y-b\sin\alpha}{m}=\frac{z-0}{n}=r \text{ (say).} \qquad .....(2)$$

Any point on (2) is $(lr+a\cos\alpha, mr+b\sin\alpha, nr)$. If it lies on (1), then

$$\frac{(lr+a\cos\alpha)^2}{a^2}+\frac{(mr+b\sin\alpha)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

or $r^2\left[\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right]+2\left(\frac{l\cos\alpha}{a}+\frac{m\sin\alpha}{b}\right)r=0 \qquad .....(3)$

If the line (2) is a generator of hyperboloid (1), it will lie wholly on (1) and for this the relation (3) will be an identity in r, for which we have

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \qquad .....(4)$$

and $l\cos\alpha/a+m\sin\alpha/b=0$

or $$\frac{l}{a\sin\alpha}=\frac{m}{b\cos\alpha}=k(say). \qquad .....(5)$$

Putting for *l* and *m* from (5) in (4), we have

$$k^2(\sin^2\alpha+\cos^2\alpha)-n^2/c^2=0 \Rightarrow n=\pm ck$$

$$\Rightarrow k=\pm\frac{n}{c} \qquad .....(6)$$

From (5) and (6), we get

$$\frac{l}{a\sin\alpha}=\frac{m}{-b\cos\alpha}=\frac{n}{\pm c}$$

Putting for *l,m,n* in (2), the required equations of the generators are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}$$

Ex.2 Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$, which pass through the point (2,3,–4).

Sol. The given hyperboloid is $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ .....(1)

Let *l,m,n* be the d.c.'s of the generator. Then, the equation of the generator through (2,3,–4) are

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r \text{ (say)} \qquad .....(2)$$

Any point on (2) is $(lr+2, mr+3, nr-4)$. If it lies on (1), then

$$\frac{1}{4}(lr+2)^2+\frac{1}{9}(mr+3)^2-\frac{1}{16}(nr-4)^2=1$$

or $$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2\left(\frac{l}{2}+\frac{m}{3}+\frac{n}{4}\right)=0 \quad .....(3)$$

If the line (2) is a generator of (1) then (3) will be an identity in r, for which we have

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \qquad .....(4)$$

and $$\frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0. \qquad .....(5)$$

Eliminating n, between (4) and (5), we have

$$\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0 \Rightarrow lm=0$$

$\Rightarrow$ either $l = 0$ or $m = 0$

When $l = 0$, then from (5), we have

$\frac{m}{3}+\frac{n}{4}=0$ or $m/3=n/-4.$

Thus, the d.c.'s of one generator are

$l/0=m/3=n/-4$

When $m = 0$, then from (5)

$\frac{l}{2}+\frac{n}{4}=0 \Rightarrow \frac{l}{1}=\frac{n}{-2}$

So the d.c.'s of the other generator are given by

$l/1=m/0=n/-2$ .....(7)

Hence the equation of the two generators through the point (2,3,–4) are given by

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4}$$

and $$\frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2} \qquad \text{[from (2)]}$$

CHECK YOUR PROGRESS :

Q.1 Find the equation of generating lines of the hyperboloid $yz + 2zx + 3xy + 6 = 0$ which passes through the point (–1, 0, 3).

[Ans. $\frac{x+1}{0} = \frac{y-0}{1} = \frac{z-3}{0}, \frac{x+1}{1} = \frac{y-0}{-1} = \frac{z-3}{3}$]

Q.2 Find the equation of generating lines of the one sheet hyperboloid

$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$ which passes through the point (2,–1,4/3)

[Ans. $\frac{x-2}{0} = \frac{y+1}{3} = \frac{z-4/3}{-4}, \frac{x-2}{3} = \frac{y+1}{6} = \frac{z-4/3}{10}$]

5.4 Confocal Conics

5.4.1 Definition: The conicoid whose principal section are confocal conics are called confocal conicoids. Thus

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1 \qquad .....(1)$$

represent the general equation of a system of confocal conicoid to the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad .....(2)$$

where λ is a parameter.

Special cases : When a>b>c and $-\infty < \lambda < \infty$

Case 1: When $\lambda < 0$ then the surface (1) is an ellipsoid.

Case 2: When $\lambda \to -\infty$ then we have sphere of radius infinity.

Case 3: When $\lambda > 0$ and $\lambda < C^2$, surface is an ellipsoid but when $\lambda \to c^2$ from the left, ellipsoid tends to ellipse.

$$\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1, z = 0 \qquad .....(3)$$

In *xy*-plane

Case 4: When $c^2 < \lambda < b^2$ then surface is a hyperbolid of one sheet, since $c^2 - \lambda < 0$.

Case 5: When $\lambda \to c^2$(from the right) the surface becomes ellipse (3) while $\lambda \to b^2$ (from the left) it becomes hyperbola.

$$y = 0, \frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2 - c^2} = 1 \qquad .....(4)$$

in *zx*-plane.

Case 6: When $b^2 < \lambda < a^2$, then surface is a hyperboloid of two sheets, since $b^2 - \lambda < 0, c^2 - \lambda < 0$. While $\lambda \to b^2$ it becomes hyperbola.

Case 7: When $\lambda \to a^2, b^2 < \lambda < a^2,$ (from the left), surface reduce to the imaginary ellipse.

$$x = 0, \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} = -1$$

Case 8: When $\lambda > a^2$, the surface is always imaginary.

5.4.2 Confocal through a given point

Let the given conicoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ (a>b>c)} \qquad ....(1)$$

and let the equation of a conicoid confocal to (1) be

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \qquad .....(2)$$

Let the confocal conicoid (2) passed through a point P(α,β,γ) say, then we have

$$\frac{\alpha^2}{a^2 - \lambda} + \frac{\beta^2}{b^2 - \lambda} + \frac{\gamma^2}{c^2 - \lambda} = 1$$

This is a cubic equation in λ, so it gives three values of λ which indicate that three confocal conicoid passes through point P. Let the values of λ be λ_1, λ_2, λ_3 in such a way that

$a^2 > \lambda_1 > b^2 > \lambda_2 > c^2 > \lambda_3$, then

Case 1: If $\lambda = \lambda_3$ the surface is an ellipsoid.

Case 2: If $\lambda = \lambda_2$ the surface is a hyperboloid of one sheet.

Case 3: If $\lambda = \lambda_1$ the surface is a hyperboloid of two sheets.

5.4.3 Condition of confocal touching to a given plane

Let the given plane be

$$lx + my + nz = p \qquad \text{.....(1)}$$

and the given confocal conicoid be

$\dfrac{x^2}{a^2-\lambda} + \dfrac{y^2}{b^2-\lambda} + \dfrac{z^2}{c^2-\lambda} = 1$, then the required condition of tangency is given by

$$l^2(a^2-\lambda) + m^2(b^2-\lambda) + n^2(c^2-\lambda) = p^2$$

5.4.4 Confocal Cut at Right Angles.

Let $P(x_1, y_1, z_1)$ be a common point of the confocals.

$$\frac{x^2}{a^2-\lambda_1} + \frac{y^2}{b^2+\lambda_1} + \frac{z^2}{c^2+\lambda_1} = 1 \qquad \text{..... (1)}$$

and $\dfrac{x^2}{a^2+\lambda_2} + \dfrac{y^2}{b^2+\lambda_2} + \dfrac{z^2}{c^2+\lambda_2} = 1$ .....(2)

Substituting $(x_{1,} y_1, z_1)$ in both equations (1) and (2), and substracting, we get

$$\left(\frac{1}{d^2+\lambda_1} - \frac{1}{a^2+\lambda_2}\right)x_1^2 + \left(\frac{1}{b^2+\lambda_1} - \frac{1}{b^2+\lambda_2}\right)y_1^2 + \left(\frac{1}{c^2+\lambda_1} - \frac{1}{c^2+\lambda_2}\right)z_1^2 = 0$$

or $(\lambda_2 - \lambda_1)\left[\dfrac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \dfrac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)} + \dfrac{z_1^2}{(c^2+\lambda_1)(c^2+\lambda_2)}\right] = 0$

or $\dfrac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \dfrac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)} + \dfrac{z_1^2}{(c^2+\lambda_1)(c^2+\lambda_2)} = 0$ .....(3)

$[\because \lambda_2 \neq \lambda_1]$

The equation of tangent planes at the common point (x_1, y_1, z_1) of the confocals (1) and (2) are respectively.

$$\frac{xx_1}{a^2+\lambda_1} + \frac{yy_1}{b^2+\lambda_1} + \frac{zz_1}{c^2+\lambda_1} = 1 \qquad \text{.....(4)}$$

and $$\frac{xx_1}{a^2+\lambda_2}+\frac{yy_1}{b^2+\lambda_2}+\frac{zz_1}{c^2+\lambda_2}=1 \qquad .....(5)$$

If planes (4) and (5) cut one other at right angle, then

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}+\frac{z_1^2}{(c^2+\lambda_1)(c^2+\lambda_2)}=0$$

which in same as (3). Thus the tangent planes at ($x_1, y_{1,} z_1$) to the confocals are at right angles.

Hence two confocal coincoids cut one another at right angles at their common points.

Cor. The tangent planes at a point, to the three confocals which pass through it, are mutually at right angles.

5.4.5 Confocals Through a Point on a Conicoid.

Let $P(x_1, y_1, z_1)$ be any point on the conicoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad .....(1)$$

If P be on the confocal whose parameter is λ, we have

$$\frac{x_1^2}{a^2-\lambda}+\frac{y_1^2}{b^2-\lambda}+\frac{z_1^2}{c^2-\lambda}=1.$$

Then the parameters of the confocals through P, are given by the following two equations:

$$\frac{x_1^2}{a^2-\lambda}+\frac{y_1^2}{b^2-\lambda}+\frac{z_1^2}{c^2-\lambda}=1.=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}$$

i.e., $$\left(\frac{1}{a^2-\lambda}-\frac{1}{a^2}\right)x_1^2+\left(\frac{1}{b^2-\lambda}-\frac{1}{b^2}\right)y_1^2+\left(\frac{1}{c^2-\lambda}-\frac{1}{c^2}\right)z_1^2=0$$

i.e., $$\lambda\left[\frac{x_1^2}{a^2(a^2-\lambda)}+\frac{y_1^2}{b^2(b^2-\lambda)}+\frac{z_1^2}{c^2(c^2-\lambda)}\right]=0$$

i.e., $$\frac{x_1^2}{a^2(a^2-\lambda)}+\frac{y_1^2}{b^2(b^2-\lambda)}+\frac{z_1^2}{c^2(c^2-\lambda)}=0 \qquad .....(2)$$

$[\because \lambda_2 \neq 0]$

Now the equation of the central section of the given conicoid parallel to the tangent plane at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad .....(3)$$

and the squares of the semi-axes of this section are given by

$$\frac{x_1^2}{a^2(a^2-r^2)}+\frac{y_1^2}{b^2(b^2-r^2)}+\frac{z_1^2}{c^2(c^2-r^2)}=0$$, which is same as (3).

Therefore the values of λ are the squares of the semi-axes of this section.

Again the direction cosines *l,m,n* of the semi axis of length *r* are given by

$$\frac{l}{x_1/(a^2-r^2)}=\frac{m}{y_1/(b^2-r^2)}=\frac{n}{z_1/(c^2-r^2)}$$, so that the axis is parallel to the normal at $P(x_1, y_1, z_1)$ to the confocal conicoid.

It follows that the parameters of two confocals through any point *P* of a conicoid are equal to the squares of the semi-axes of the central section of the conicoid which is parallel to the tangent plane at *P*, and the normals at *P* to the confocals are parallel to the axes of that section.

SOLVED EXAMPLE

Ex.1 Show that the product of the eccentricities of the focal conics of an ellipsoid in unity.

Sol. Let the given ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1(a>b>c) \qquad .....(1)$$

and let it confocal conicoids be

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \qquad .....(2)$$

λ being a parameter. Then the equation of confocal ellipse and confocal hyperbola of the confocal conicoides (2) are respectively:

$$z=0, \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1 \qquad .....(3)$$

and $y = 0, \dfrac{x^2}{a^2 - b^2} - \dfrac{z^2}{b^2 - c^2} = 1$ .....(4)

Let e_1 and e_2 be eccentricities of the confocals (3) and (4) respectively, then

$$\frac{b^2 - c^2}{a^2 - c^2} = 1 - e_1^2, \text{ i.e., } e_1 = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

and $\dfrac{b^2 - c^2}{a^2 - c^2} = e_2^2 - 1, \text{ i.e., } e_2 = \sqrt{\dfrac{a^2 - c^2}{a^2 - b^2}}$

Hence the product of eccentricities

$= e_1.e_2$

$$= \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \cdot \sqrt{\frac{a^2 - c^2}{a^2 - b^2}} = 1$$

CHECK YOUR PROGRESS :

Q.1 Show that two confocal paraboloids cut everywhere at right angles.

Q.2 Prove that three paraboloids confocal with a given paraboloid pass through a given point, of which two are elliptic and one is hyperbolic.

Q.3 Prove that the locus of plane section of a confocal conicoid of the given conicoid is a straight line.

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